From Fourier Series to Analysis of Non-stationary Signals – III

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Contents

Review vectors

Recall vectors in n-dimensional space $\mathbb{R}^n$. Each such vector $\mathbf{u}$ can be uniquely represented as a linear combination of n unit vectors $\mathbf{e_1}, ..., \mathbf{e_n}$:

$$\mathbf{u} = \alpha_1 \mathbf{e_1} + \alpha_2 \mathbf{e_2} + \ldots + \alpha_n \mathbf{e_n},$$

where α_i are real numbers. These can be computed using the scalar (or inner) product as follows:

$$\alpha_i = (\mathbf{u}, \mathbf{e_i}) \text{ for } i = 1, \ldots, n$$

Review vectors

Vectors $\mathbf{e}_i$ are orthonormal e.g. :

- normalized $\mathbf{e}_i.\mathbf{e}_i = \|\mathbf{e}_i\|^2 = 1$ for all i
- orthogonal $\mathbf{e}_i.\mathbf{e}_j = (\mathbf{e}_i, \mathbf{e}_j) = 0$ for $i \neq j$

Example

Addition Draw addition of two vectors in two dimensional space $\mathbb{R}^2$, $\mathbf{u} = 3\mathbf{e}_1 + 4\mathbf{e}_2$ and $\mathbf{v} = -2\mathbf{e}_1 + 3\mathbf{e}_2$ and make them normalized.

Review vectors

Vectors are objects that can be added together and multiplied by scalars - vector space:

- if $\mathbf{u} = \sum_{i=1}^{n} \alpha_i \mathbf{e}_i$ and $\mathbf{v} = \sum_{i=1}^{n} \beta_i \mathbf{e}_i \Rightarrow$

$$\mathbf{u} + \mathbf{v} = \sum_{i=1}^{n} (\alpha_i + \beta_i) \mathbf{e}_i$$

- if $\mathbf{u} = \sum_{i=1}^{n} \alpha_i \mathbf{e}_i$ and λ is scalar $\Rightarrow$

$$\lambda \mathbf{u} = \sum_{i=1}^{n} \lambda \alpha_i \mathbf{e}_i$$

Vector space of continuous-time signals

We have already studied the space of continuous-time signals. We can easily verify:

- we can form the sum of any two signals $x_1(t)$ and $x_2(t)$ to obtain another signal
$$x(t) = x_1(t) + x_2(t)$$
- we can multiply any signal $x(t)$ by a constant λ to obtain another signal
$$y(t) = \lambda x(t)$$

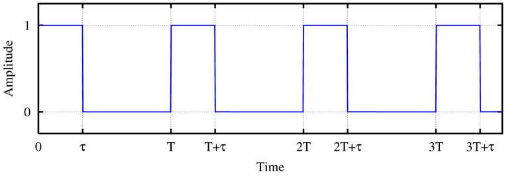
Unlike the n dimensional space $\mathbb{R}^n$, the vector space of all continuous-time signals is infinite-dimensional.

Vector space of periodic signals

Consider now periodic signals; any such signal $x(t)$ satisfies periodicity condition:

$$x(t + T) = x(t) \text{ for all } t$$

and for given period T.



Vector space of periodic signals

It is easy to see that periodic signals form a vector space:

- if $x_1(t)$ and $x_2(t)$ are periodic, then

$$x(t+T) = x_1(t+T) + x_2(t+T) = x_1(t) + x_2(t)$$

  is also periodic with the same period T

- if $x_1(t)$ is periodic and β is scalar, then

$$\beta x(t+T) = \beta x(t)$$

  is scaled version of $x(t)$ being also periodic with period T

Vector space of periodic signals

If we impose even more conditions on periodic signals - the Dirichlet conditions, which hold for all signals encountered in practice, then we can represent signals as infinite linear combinations of orthogonal and normalized vectors.

- A function satisfying Dirichlet conditions must have right and left limits at each point of discontinuity:

$$x(t+) = \lim_{\tau \to t+} x(\tau) \text{ and } x(t-) = \lim_{\tau \to t-} x(\tau)$$

- The Dirichlet theorem says in particular that the Fourier series for $x(t)$ converges and is equal to $x(t) = \frac{x(t+)+x(t-)}{2}$ wherever $x(t)$ is continuous.

Complete orthonormal systems

First of all, we can define a scalar (or inner) product of two T-periodic signals $x_1(t)$ and $x_2(t)$ as

$$(x_1(t), x_2(t)) = \int_0^T x_1(t)x_2(t)dt$$

We can integrate over any complete period, i.e. from $-\frac{T}{2}$ to $-\frac{T}{2}$

$$(x_1(t), x_2(t)) = \int_{-\frac{T}{2}}^{\frac{T}{2}} x_1(t)x_2(t)dt.$$

Complete orthonormal systems

Then we can take any sequence of T -periodic functions $\phi_0(t), \phi_1(t), \phi_2(t), \ldots$ that are

- normalized: $(\phi_i(t), \phi_i(t)) = \|\phi_i(t)\|^2 = \int_0^T \phi_i^2(t)dt$
- orthogonal: $(\phi_i(t), \phi_j(t)) = \int_0^T \phi_i(t)\phi_j(t)dt = 0$ for $i \neq j$
- complete: if a signal $x(t)$ is such that

$$(\phi_i(t), x(t)) = \int_0^T \phi_i(t)x(t)dt = 0$$

for all k, then $x(t) = 0$

Fourier Series

Let $\{\phi_j(t)\}_{j=0}^{\infty}$ be a complete, orthonormal set of functions. Then any well-behaved T -periodic signal $x(t)$ can be uniquely represented as an infinite series

$$x(t) = \sum_{j=0}^{\infty} \alpha_j \phi_j(t)$$

This is called the Fourier series representation of $x(t)$. The scalars (numbers) α_j are called the Fourier coefficients of $x(t)$ with respect to $\{\phi_j(t)\}_{j=0}^{\infty}$ and are computed as follows:

$$\alpha_j = (\phi_j(t), x(t)) = \int_0^T \phi_j(t) x(t) dt.$$

Fourier Series

In analogy to vectors in n dimensional space, you can think of α_j as the projection of $x(t)$ in the direction of $\phi_j(t)$.
proof
To derive the formula for α_j , write

$$x(t)\phi_k(t) = \sum_{j=0}^{\infty} \alpha_j \phi_j(t)\phi_k(t)$$

and then integrate over a period

$$(\phi_k(t), x(t)) = \int_0^T \phi_k(t)x(t)dt = \int_0^T dt \sum_{j=0}^{\infty} \alpha_j \phi_j(t)\phi_k(t).$$

Fourier Series

For convergent series we can integrate term by term and

$$\int_0^T dt \sum_{j=0}^{\infty} \alpha_j \phi_j(t)\phi_k(t) = \sum_{j=0}^{\infty} \alpha_j \int_0^T \phi_j(t)\phi_k(t)dt = \sum_{j=0}^{\infty} \alpha_j \delta_{j,k} = \alpha_k$$

Here and in following evaluation we will use Kronecker delta which is defined as $\delta_{j,k} = 0$ for $j \neq k$ and $\delta_{k,k} = 1$ and which indicates that $\{\phi_j(t)\}_{j=0}^{\infty}$ form an orthonormal system of functions.

Fourier Series

It can be also proved that, as the functions $\{\phi_j(t)\}_{j=0}^{\infty}$ form a complete orthonormal system, the partial sums of the Fourier series

$$x(t) = \sum_{j=0}^{\infty} \alpha_j \phi_j(t)$$

converge to $x(t)$ in the following sense:

$$\lim_{N\to\infty} \int_0^T dt \left(x(t) - \sum_{j=0}^{N} \alpha_j \phi_j(t) \right)^2$$

Fourier Series

So, we can use (with some care for discontinuities) the partial sums

$$x(t) = \sum_{j=0}^{N} \alpha_j \phi_j(t)$$

to approximate $x(t)$.

Trigonometric Fourier Series

The sequence of T-periodic functions $\{\phi_k(t)\}_{k=0}^{\infty}$ defined by

1. $\phi_0(t) = \dfrac{1}{\sqrt{T}}$
2. $\phi_k(t) \equiv \phi_{2m}(t) = \sqrt{\dfrac{2}{T}}\cos(2m\omega_0 t)$ if $k = 2m$ is even
3. $\phi_k(t) \equiv \phi_{2m-1}(t) = \sqrt{\dfrac{2}{T}}\sin((2m-1)\omega_0 t)$ if $k = 2m-1$ is odd

is complete and orthonormal. Here $\omega_0 = \dfrac{2\pi}{T}$ is called fundamental frequency.

Trigonometric Fourier Series

Common way of writing down the trigonometric Fourier series of $x(t)$ is following:

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$$

The Fourier coefficients can be computed as follows:

1. $a_0 = \frac{1}{T} \int_0^T x(t) dt$
2. $a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t) dt$
3. $b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t) dt$

Trigonometric Fourier Series

To relate this to the orthonormal representation in terms of the $\{\phi_k(t)\}_{k=0}^{\infty}$, we note that we can write

1. $a_0 = \frac{1}{\sqrt{T}} \int_0^T x(t)\phi_0(t)dt = \frac{1}{\sqrt{T}}\alpha_0$
2. $a_k = \sqrt{\frac{2}{T}} \int_0^T x(t)\phi_{2m}(t)dt = \sqrt{\frac{2}{T}}\alpha_{2m}$
3. $b_k = \sqrt{\frac{2}{T}} \int_0^T x(t)\phi_{2m-1}(t)dt = \sqrt{\frac{2}{T}}\alpha_{2m-1}$
4. $x(t) = a_0 + \sum_{k=0}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=0}^{\infty} b_k \sin(k\omega_0 t) \equiv \sum_{k=0}^{\infty} \alpha_k \phi_k(t).$

Trigonometric Fourier Series - Symmetry

1. if $x(t)$ is an even function, i.e., $x(t) = x(-t)$ for all t, then all its sine Fourier coefficients are zero:

$$b_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\sin(k\omega_0 t)dt = 0$$

2. if $x(t)$ is an odd function, i.e., $x(t) = -x(-t)$, then all its cosine Fourier coefficients are zero:

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\cos(k\omega_0 t)dt = 0$$

Trigonometric Fourier Series

Even function

Theorem (Fourier series of even function)

Fourier series of an even function $f(t) = f(-t)$ consists of the constant and cosine terms

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t),$$

where $\omega_0 = \dfrac{2\pi}{T}$.

Trigonometric Fourier Series

Odd function

Theorem (Fourier series of odd function)

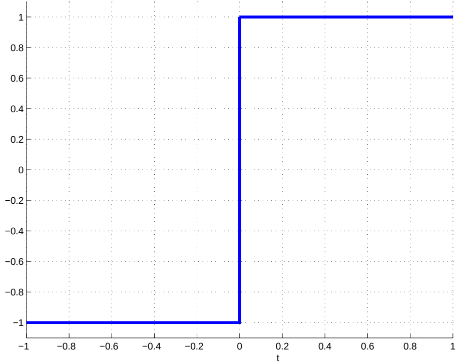
Fourier series of an odd function $f(t) = -f(-t)$ consists of the sine terms

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t),$$

where $\omega_0 = \dfrac{2\pi}{T}$.

Trigonometric Fourier Series

Example 1: Consider a periodic signal $x(t) = x(t + T)$ given by repeating the square wave



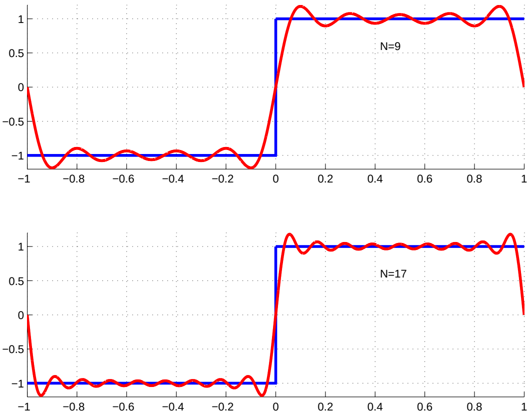
Note, that here $T = 2$!

Trigonometric Fourier Series

Solution:

1. the signal has odd symmetry $\Rightarrow$ all $a_k = 0$
2. $b_k = \frac{2}{T}\int_{-1}^{1} x(t)\sin(k\omega_0 t)dt = \frac{2}{T}\int_{-1}^{0}(-1)\sin(k\omega_0 t)dt + \frac{2}{T}\int_{0}^{1}(+1)\sin(k\omega_0 t)dt = \frac{1}{k\pi}\left[cos(k\pi t)\right]_{-1}^{0} - \frac{1}{k\pi}\left[cos(k\pi t)\right]_{0}^{1} = \frac{2}{k\pi}(1-\cos(k\pi)) = \frac{4}{k\pi}\sin^2(\frac{k\pi}{2})$
3. For $k = 2m-1$ is $b_k = \frac{4}{k\pi}\sin^2(\frac{k\pi}{2}) = \frac{4}{(2m-1)\pi}$
4. $x(t) = \sum_{m=1}^{\infty} \frac{4}{(2m-1)\pi}\sin((2m-1)\pi t)$

Partial sums $x_N(t) = \sum_{m=1}^{N} \frac{4}{(2m-1)\pi} \sin(2m-1)\pi t$



Gibbs phenomenon

The Fourier series (over/under)shoots the actual value of $x(t)$ at points of discontinuity regardless of degree N.

Complex exponentials

Another useful complete orthonormal set is accomplished by the complex exponentials:

1. $\phi_k(t) = \frac{1}{\sqrt{T}} \exp(j\,k\omega_0 t)$ for $k = \ldots -2, -1, 0, 1, 2, \ldots$
2. these functions are complex-valued, and we have to evaluate the inner product as

$$(x_1(t), x_2(t)) = \int_0^T x_1(t) x_2^*(t) dt,$$

where $x_2^*(t)$ denotes complex conjugation

Complex exponential Fourier Series

1. $(\phi_k(t), \phi_\ell(t)) = \dfrac{1}{T}\displaystyle\int_0^T \exp(j\,k\omega_0 t)\exp(-j\,\ell\omega_0 t)dt = \delta_{k,\ell}$
2. $x(t) = \displaystyle\sum_{k=-\infty}^{\infty} c_k \exp(j\,k\omega_0 t)$
3. $c_k = \dfrac{1}{T}\displaystyle\int_0^T x(t)\exp(-j\,k\omega_0 t))dt$
4. as in trigonometric case $\omega_0 = \dfrac{2\pi}{T}$

Trigonometric Fourier Series

Problem 1: Find the Fourier series representation for the half-wave rectified sinusoid.

$$f(t) = \begin{cases} \sin(\frac{2\pi t}{T}) & \text{if} \quad 0 \le t \le \frac{T}{2} \\ 0 & \text{if} \quad \frac{T}{2} \le t \le T \end{cases}$$

- Calculate the coefficients a_k and b_k using identities

$$\begin{aligned} 2\sin \ell x \sin mx &= \cos(\ell - m)x - \cos(\ell + m)x, \qquad (1) \\ 2\sin \ell x \cos mx &= \sin(\ell - m)x + \sin(\ell + m)x. \end{aligned}$$

- Plot the first 5 components of the Fourier Series using MATLAB.

Trigonometric Fourier Series

Problem 2: Find the Fourier series representation for the sawtooth $f(t) = f(t+T) = t \text{ if } -\frac{T}{2} \leq t \leq \frac{T}{2}$.

- As the function $f(t)$ is odd the coefficients $a_k = 0$. Calculate coefficients b_k.
- Plot the first 5 components of the Fourier Series using MATLAB.

Homework 3

Trigonometric Fourier series

1. Calculate the Fourier series for $f(x) = x^2$ if $-A \leq x \leq A$.
2. Compare the results for space periodicity $f(x + 2A) = f(x)$ with those obtained for time periodicity $f(t + T) = f(t)$.
3. Plot the first 5 components of the Fourier Series using MATLAB.