Novel selective transform for non-stationary signal processing

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Contents

1. Introduction
2. Basis functions for a new multispectral transformation
3. Fourier and Zolotarev Transforms
4. Applications

Non-stationary signals

- Non-stationary signals occur almost everywhere, in broadcasting, in seismic physics, and in numerous other interdisciplinary fields such as multimedia, telecommunications, radar, sonar, vibration analysis, speech processing, and medical diagnosis (as EEG, or ECG signals). click to play
- Scientists have been studying the time-frequency phenomena of such signals for a considerable period.

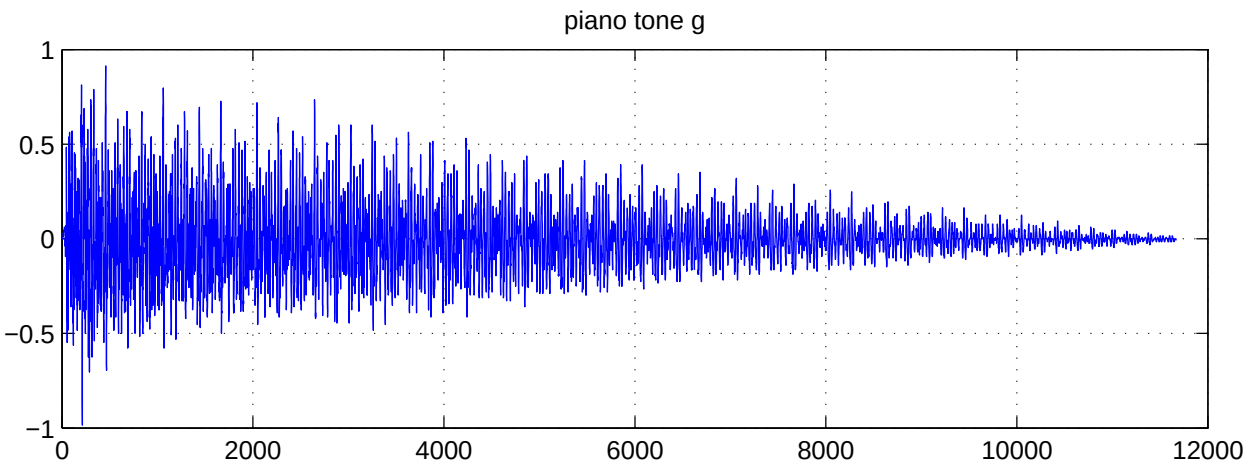


Figure: A non-stationary signal

Non-stationary signals

- There are several approaches available for addressing non-stationary signals, including the short time Fourier Transform (STFT), the fractional Fourier Transform (FRFT), the Wavelet Transform (WT), and the Hilbert-Huang Transform (HHT).
- A large group of non-linear multiresolution transforms is also available...
- ... but very often they suffer from the difficult interpretation
- we have introduced a novel tool - Discrete Zolotarev Transform (DZT) - 2009

Discrete Zolotarev Transform

Basis functions for Discrete Zolotarev Transform are derived from the family of Zolotarev polynomials, which are iso-extremal within two disjoint intervals

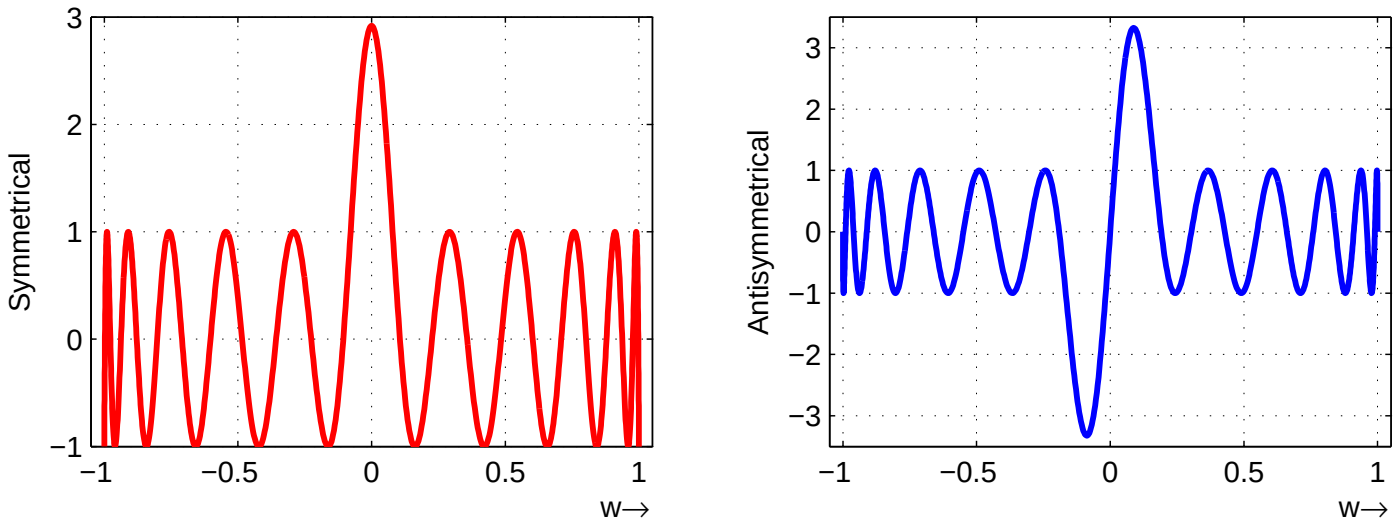


Figure: Symmetrical polynomial $Z_{11,11}(w, 0.52)$ and antisymmetrical iso-extremal function based on polynomial $\mathcal{S}_{10,1}(w, 0.995)$

Discrete Zolotarev Transform

R. Spetik, Discrete Zolotarev Transform, Doctoral thesis, Czech Technical University in Prague, 126 p., 2009.

- (DZT) is signal dependent transform, representing clearly the non-stationary segments
- (DZT) partially avoids Heisenberg principle of uncertainty, reduces spectral leakages
- (DZT)provides a non-stationarity measure

	STFT	Wavelets	Hilbert-Huang	DZT
inversion	yes	yes, but...	no	yes
time resolution	limited	yes	yes	yes
frequency resolution	yes	bad or n/a	floating components	yes

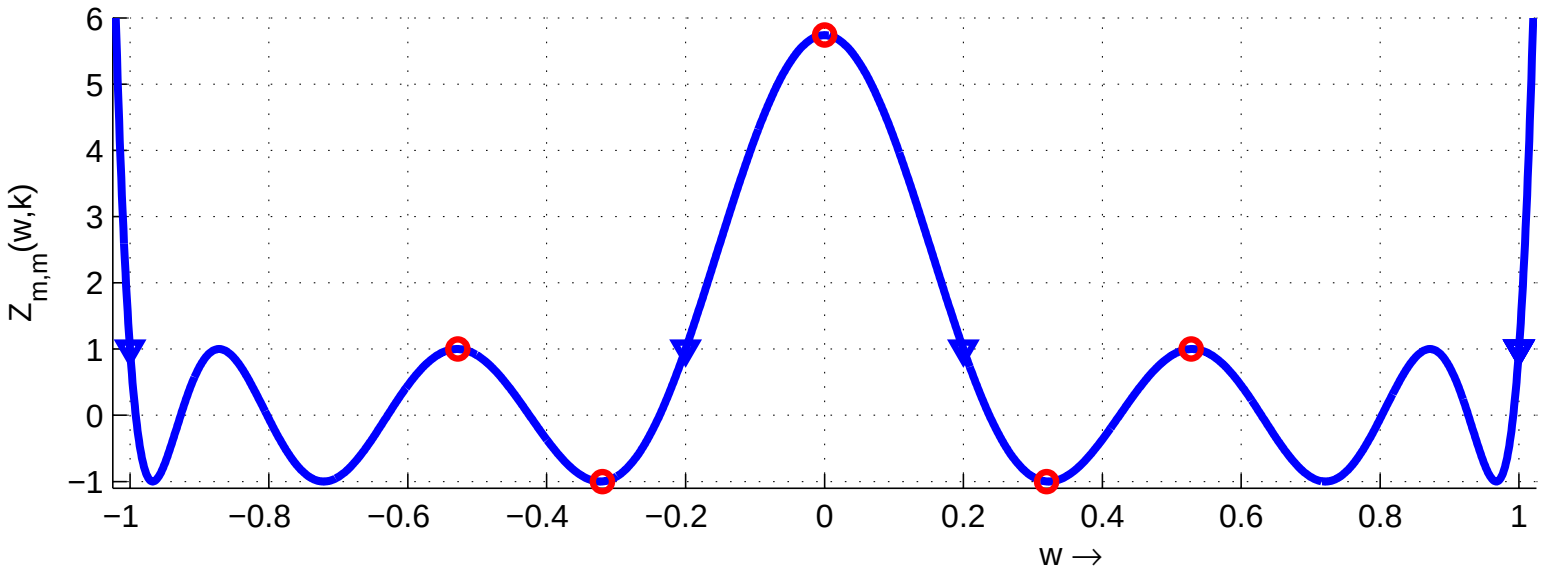
Discrete Zolotarev Transform

In this lecture

- few words on mathematical problems concerning the iso-extremal polynomials
- construction of basis functions and definition of the Discrete Zolotarev Transform
- numerical experiments and applications to several signals

Symmetrical Zolotarev polynomials - approximation

$$(1-w^2)(w^2-\kappa'^2)\left(\frac{dZ_{m,m}(w)}{dw}\right)^2 = 4\,m^2w^2\,(1-Z_{m,m}^2(w))$$



The second order differential equation

$$(1-w^2)(w^2-\kappa'^2)\left(\frac{dy(w)}{dw}\right)^2 = 4m^2w^2\left(1-y^2(w)\right)$$

$$\Downarrow$$

$$w(w^2-\kappa'^2)\left[(1-w^2)\frac{d^2y}{dw^2} - w\frac{dy}{dw}\right] + \kappa'^2(1-w^2)\frac{dy}{dw} + 4m^2w^3y = 0$$

$$y(w) \equiv T_m\left(\frac{2w^2-1-\kappa'^2}{1-\kappa'^2}\right) = \sum_{\ell=0}^{m} b(2\ell)w^{2\ell}$$

.. is a polynomial of variable x

An alternative representation

$$y(w) \equiv T_m\left(\frac{2w^2 - 1 - \kappa'^2}{1 - \kappa'^2}\right) = \sum_{\ell=0}^{m} b(2\ell) w^{2\ell} = \sum_{\ell=0}^{m} a(2\ell) T_{2\ell}(w)$$

.. is developed in terms of Chebyshev polynomials

Inserting $y(w) = \sum_{\ell=0}^{m} a(2\ell) T_{2\ell}(w)$ in differential equation

$$w(w^2 - \kappa'^2)\left[(1 - w^2)\frac{d^2y}{dw^2} - w\frac{dy}{dw}\right] + \kappa'^2(1 - w^2)\frac{dy}{dw} + 4m^2w^3y = 0$$

we obtain...

Recursive algorithm for the coefficients $a(2\ell)$

$$
\begin{aligned}
&\textit{given} && m, \kappa' \\
&\textit{initialisation} && a(2m) = \frac{1}{(1-k'^2)^m} \\
& && a(2m+2) = a(2m+4) = 0 \\
&\textit{recursive body} && \\
&(\textit{for} && \ell = m-1 \quad \textit{to} \quad 0 \\
& && -\left[m^2-\ell^2\right] a(2\ell) = \\
& && +\left[3(m^2-(\ell+1)^2)+(2\ell+2)(2\ell+1)\kappa'^2\right] a(2\ell+2) \\
& && +\left[3(m^2-(\ell+2)^2)+(2\ell+4)(2\ell+5)\kappa'^2\right] a(2\ell+4) \\
& && +\left[m^2-(\ell+3)^2\right] a(2\ell+6) \\
&\textit{end}\ ) &&
\end{aligned}
$$

Basis function Zcos(ϕ)

A symmetrical Zolotarev polynomial of the 1st kind

$$
\begin{aligned}
(-1)^m Z_{m,m}(w,\kappa) &= \sum_{\mu=0}^{2m} a(\mu) T_\mu(w) = \sum_{\ell=0}^{m} b(2\ell) w^{2k} \\
&= \sum_{\ell=0}^{m} a(2\ell) T_{2\ell}(w).
\end{aligned}
$$

provides the basis Zolotarev cosine function

$$
\mathrm{Zcos}(\phi) = \sum_{\ell=0}^{m} a(2\ell)\ \cos(2\ell\phi).
$$

Basis function Zcos(ϕ)

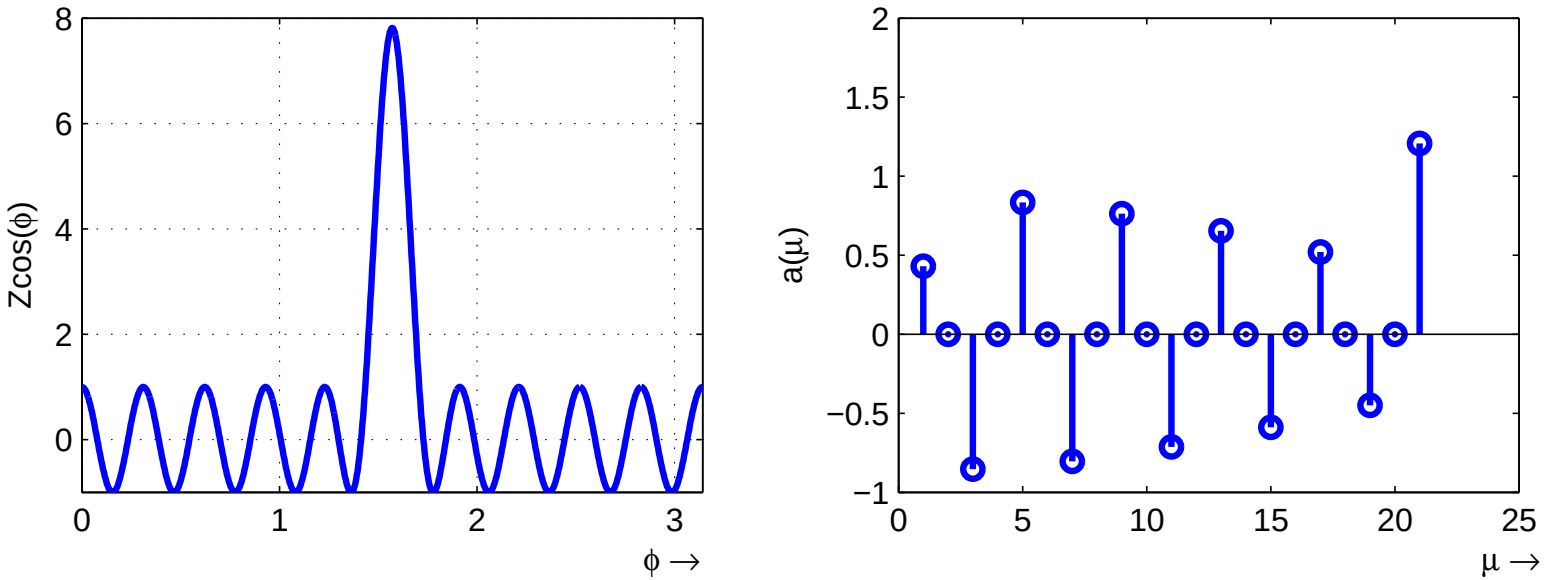


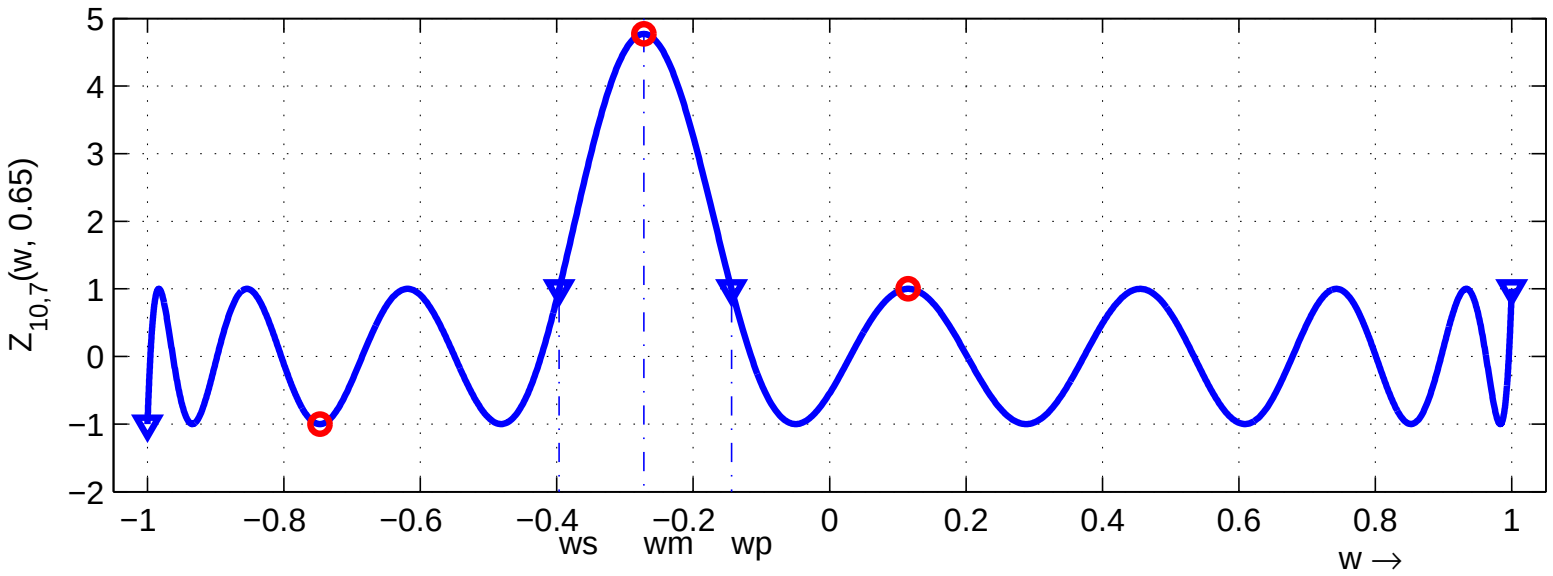
Figure: Zolotarev cosine function Zcos(ϕ) and its coefficients

Linear differential equation

For a general Zolotarev polynomial of the 1st kind $Z_{p,q}(w)$ the second order linear differential equation is valid

$$\sqrt{f(w)}\frac{1}{\sqrt{g(w)}}\frac{d}{dw}\left[\sqrt{f(w)}\frac{1}{\sqrt{g(w)}}\frac{d}{dw}Z_{p,q}(w)\right]+n^2Z_{p,q}(w)=0 \quad (1)$$

where $f(w)=(1-w^2)(w-w_p)(w-w_s)$, and $g(w)=(w-w_m)^2$.



Basis function Zsin(ϕ)

If we look for the second solution of eq. (1) having zeros at $w = \pm 1$, by substitution $\sqrt{1-w^2}\mathcal{S}_{p,q}(w,\kappa)$ we arrive to the differential equation

$$\sqrt{\frac{f(w)}{1-w^2}}\frac{1}{\sqrt{g(w)}}\frac{d}{dw}\left[\sqrt{f(w)}\frac{1}{\sqrt{g(w)}}\frac{d}{dw}\sqrt{1-w^2}\mathcal{S}_{p,q}(w,\kappa)\right]+ \quad (2)$$
$$+n^2\mathcal{S}_{p,q}(w,\kappa)=0.$$

We have developed the backward recursion for coefficients $a_+(\ell)$ of Zolotarev polynomial of the fourth kind $\mathcal{S}_{p,q}(w,\kappa)$

$$\mathcal{S}_{p,q}(w,\kappa)=(-1)^p\sum_{\ell=0}^{n}a_+(\ell)U_\ell(w).$$

Recursive algorithm for the coefficients $a_+(\ell)$

given p, q, κ

initialisation

$$n = p + q, \quad u_0 = \frac{2p+1}{2n+2}\mathbf{K}(\kappa)$$

$$w_p = 2\,\mathrm{cd}^2(u_0|\kappa) - 1, \quad w_s = 2\,\mathrm{cn}^2(u_0|\kappa) - 1$$

$$w_q = \frac{w_p + w_s}{2}, \quad w_m = w_s + 2\frac{\mathrm{sn}(u_0|\kappa)\mathrm{cn}(u_0|\kappa)}{\mathrm{dn}(u_0|\kappa)}\mathrm{Z}(u_0|\kappa)$$

$$\alpha(n) = 1$$

$$\alpha(n+i) = 0 \quad \textit{for} \quad i = 1, 2, ..., 5$$

...coefficients $a_+(\ell)$ ... cont.

(*for* $m = n+2$ *to* 3)

$$8c(1) = n(n+2) - (m+3)(m+5)$$

$$4c(2) = 3w_m[n(n+2) - (m+2)(m+4)] + (m+3)(2m+7)(w_m - w_q)$$

$$8c(3) = 3[n(n+2) - (m+1)(m+3)] + 12w_m[(n+1)^2 w_m - (m+2)^2 w_q] - 4(m+2)(m+3)(w_p w_s - w_m w_q)$$

$$2c(4) = 3[(n+1)^2 w_m - (m+1)^2 w_q] - (m+1)^2(w_m - w_q) + 2w_m[(n+1)^2 w_m^2 - (m+1)^2 w_p w_s]$$

$$8c(5) = 3[n(n+2) - (m-1)(m+1)] + 12w_m[(n+1)^2 w_m - m^2 w_q] - 4m(m-1)(w_p w_s - w_m w_q)$$

$$4c(6) = 3w_m[n(n+2) - (m-2)m] + (m-1)(2m-3)(w_m - w_q)$$

$$8c(7) = n(n+2) - (m-3)(m-1)$$

$$\alpha(m-3) = \frac{1}{c(7)} \sum_{\ell=1}^{6} (-1)^\ell c(\ell)\alpha(m+4-\ell)$$

(*end loop on* m)

coefficients $a_+(\ell)$... cont.

$$\textit{normalisation} \quad s(n) = \sum_{k=0}^{n} \alpha(k)(k+1)$$

$$(\textit{for} \quad \ell = 0 \quad \textit{to} \quad n)$$

$$a_+(\ell) = (-1)^p (n+1) \frac{\alpha(\ell)}{s(n)}$$

$$(\textit{end} \quad \textit{loop} \quad \textit{on} \quad \ell)$$

Basis function Zsin(ϕ)

For $p = 2m$ and $q = 1$ the function $\sqrt{1-w^2}\mathcal{S}_{2m,1}(w,\kappa)$ is an odd function and has following form (3)

$$\sqrt{1-w^2}\mathcal{S}_{2m,1}(w,\kappa) = \sqrt{1-w^2}\sum_{\mu=1}^{2m+1} a_+(\mu)U_\mu(w) =$$

$$= \sqrt{1-w^2}\sum_{\ell=1}^{m+1} a_+(2\ell-1)U_{2\ell-1}(w).$$

It provides with $w = \cos\phi$ definition of Zolotarev sine function

$$\mathrm{Zsin}(\phi) = \sum_{\ell=1}^{m+1} a_+(2\ell-1)\,\sin(2\ell\phi).$$

Basis function Zsin(ϕ)

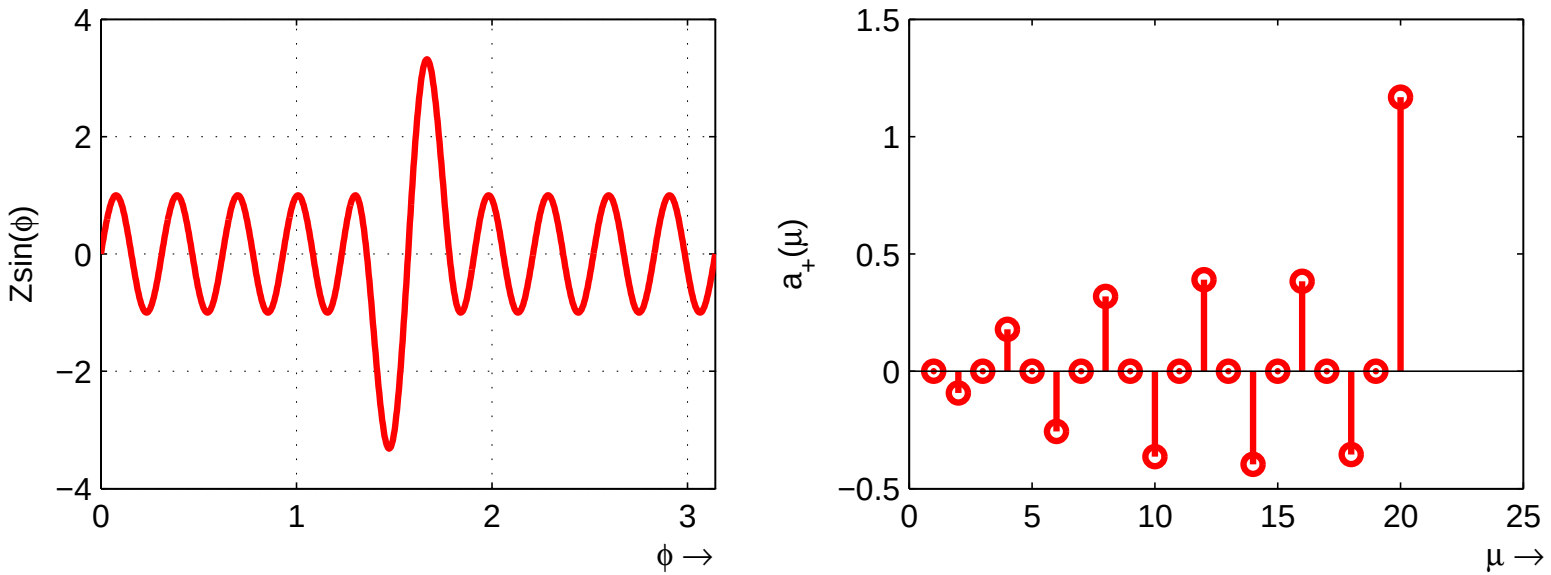


Figure: Zolotarev sine functions Zsin(ϕ) and its coefficients

Application of Zolotarev polynomials in Signal Processing

- in Digital filter design
  - Vlcek M., Zahradnik P.: Almost Equiripple Low-Pass FIR Filters, *Circuits, Systems, and Signal Processing*, Birkhauser , April 2013, vol. 32, no. 2, pp. 743-757.
  - Zahradnik P., Vlcek M.: Perfect Decomposition Narrow-Band FIR Filter Banks, *IEEE Transactions on Circuits and Systems II: Express Briefs*, October 2012, vol. 59 , no. 11, pp. 805 - 809.
  - Zahradnik P., Vlcek M.: Equiripple Approximation of Half-Band FIR Filters, *IEEE Trans. on Circuits and Systems*, vol. 56, no. 12, December 2009, pp. 941-945.
- recent research in Multispectral Analysis of Non-stationary Signals - following slides

Fourier and Zolotarev basis functions

For $\exp(i2\pi\ell t) = \cos(2\pi\ell t) + i\ \sin(2\pi\ell t)$ a band-limited signal $s(t)$ can be expressed as

$$s(t) = \sum_{\ell=-N}^{N} S(\ell) \exp(i2\pi\ell t),$$

where the coefficients $S(\ell)$ represent the frequency spectrum of the signal

$$S(\ell) = \langle \exp(-i2\pi\ell t), s(t) \rangle = 0,\ \forall |\ell| > N. \qquad (3)$$

Fourier and Zolotarev basis functions

For the basis functions $\mathrm{Zexp}(i2\pi\ell t) = \mathrm{Zcos}(2\pi\ell t) + i\,\mathrm{Zsin}(2\pi\ell t)$ a band-limited signal $s(t)$ can be expressed as

$$s(t) = \sum_{\ell=-N}^{N} S_Z(\ell)\mathrm{Cexp}(i2\pi\ell t).$$

Functions $\mathrm{Cexp}(i2\pi\ell t)$ form a biorthogonal basis to $\mathrm{Zexp}(i2\pi\ell t)$

$$\left(\{\mathrm{Zexp}(i2\pi\ell t)\}_{\ell=-N}^{N}\right)^{-1} = \{\mathrm{Cexp}(i2\pi\ell t)\}_{\ell=-N}^{N}$$

Coefficients $S_Z(\ell)$ define the Zolotarev spectrum

$$S_Z(\ell) = \langle \mathrm{Zexp}(-i2\pi\ell t), s(t)\rangle = 0,\ \forall|\ell| > N.$$

Discrete Zolotarev Transform

The Zolotarev basis can be transferred from continuous time domain into the discrete time domain

$$\text{Zexp}\left(i\frac{2\pi\ell n}{N}\right) = \text{Zcos}\left(\frac{2\pi\ell n}{N}\right) + i\,\text{Zsin}\left(\frac{2\pi\ell n}{N}\right)$$

Then the band-limited signal s(n) is expressed as

$$s(n) = \sum_{\ell=-N}^{N} S_Z(\ell)\text{Cexp}\left(i\frac{2\pi\ell n}{N}\right).$$

Zolotarev spectrum is defined as

$$S_Z(\ell) = \langle \text{Zexp}\left(-i\frac{2\pi\ell n}{N}\right), s(n)\rangle = 0,\ \forall|\ell| > N.$$

Discrete Zolotarev Transform

Using a matrix notion, the band-limited signal s(n) can be written as

$$\mathbf{s} = \mathbf{W_C} \cdot \mathbf{S_Z},$$

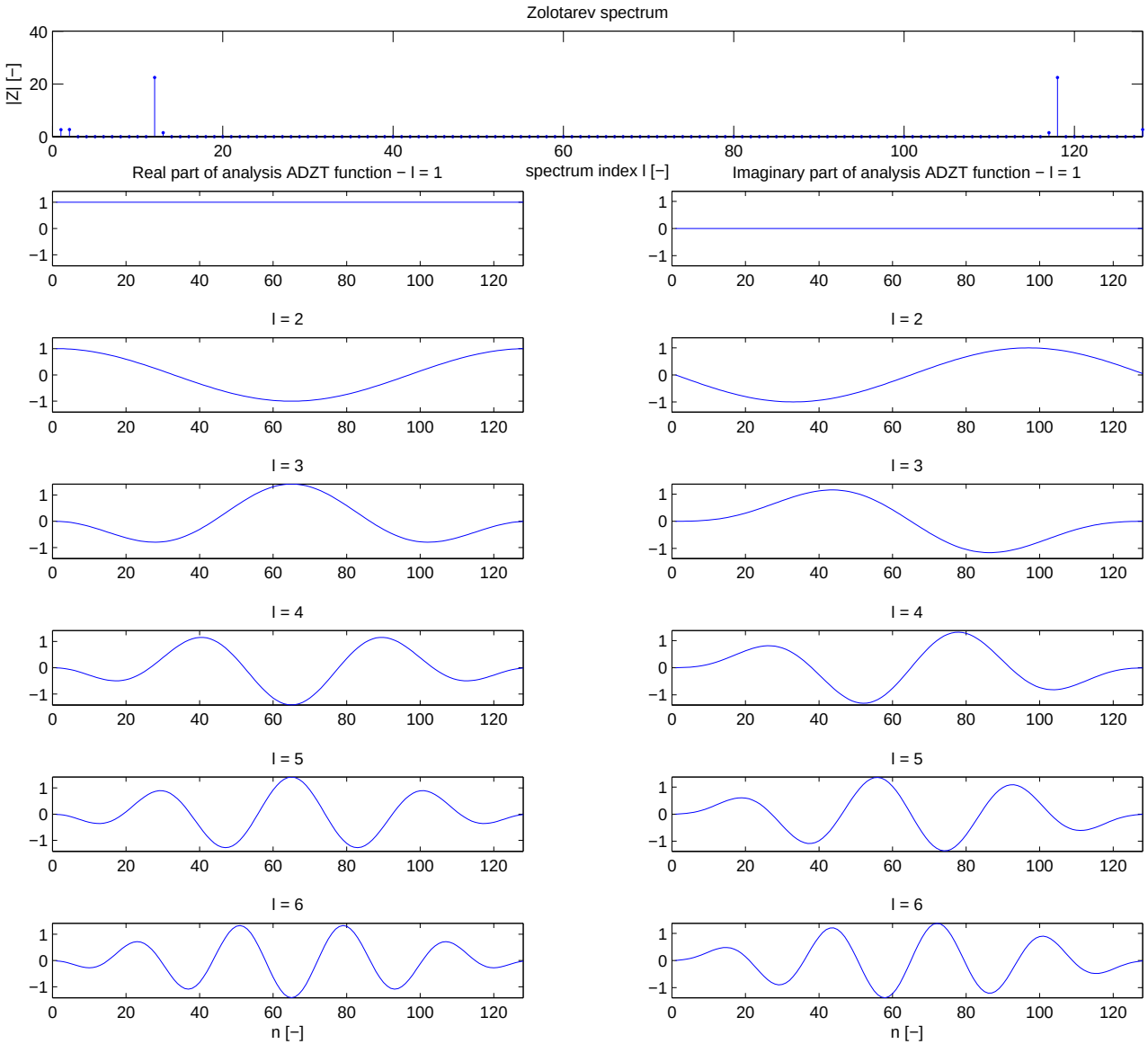
where $\mathbf{W_C}$ contains the Zolotarev synthesis functions and $\mathbf{S_Z}$ consists of Zolotarev spectrum coefficients. Thus Zolotarev spectra are given as

$$\mathbf{S_Z} = \mathbf{W_Z} \cdot \mathbf{s},$$

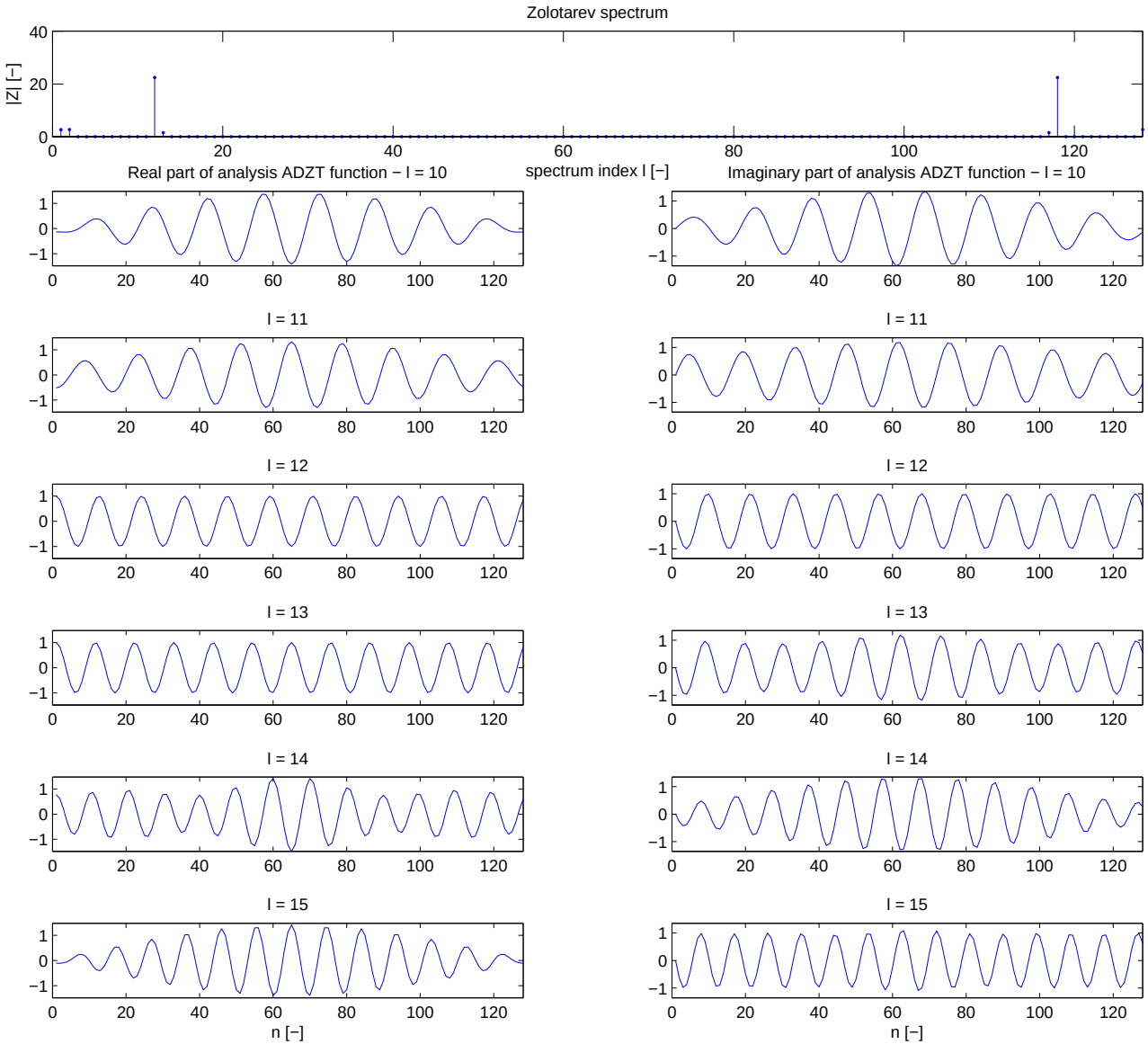
where $\mathbf{W_Z}$ contains the Zolotarev analysis functions and it is biorthogonal

$$\mathbf{W_Z} \cdot \mathbf{W_C} = \mathbf{E}.$$

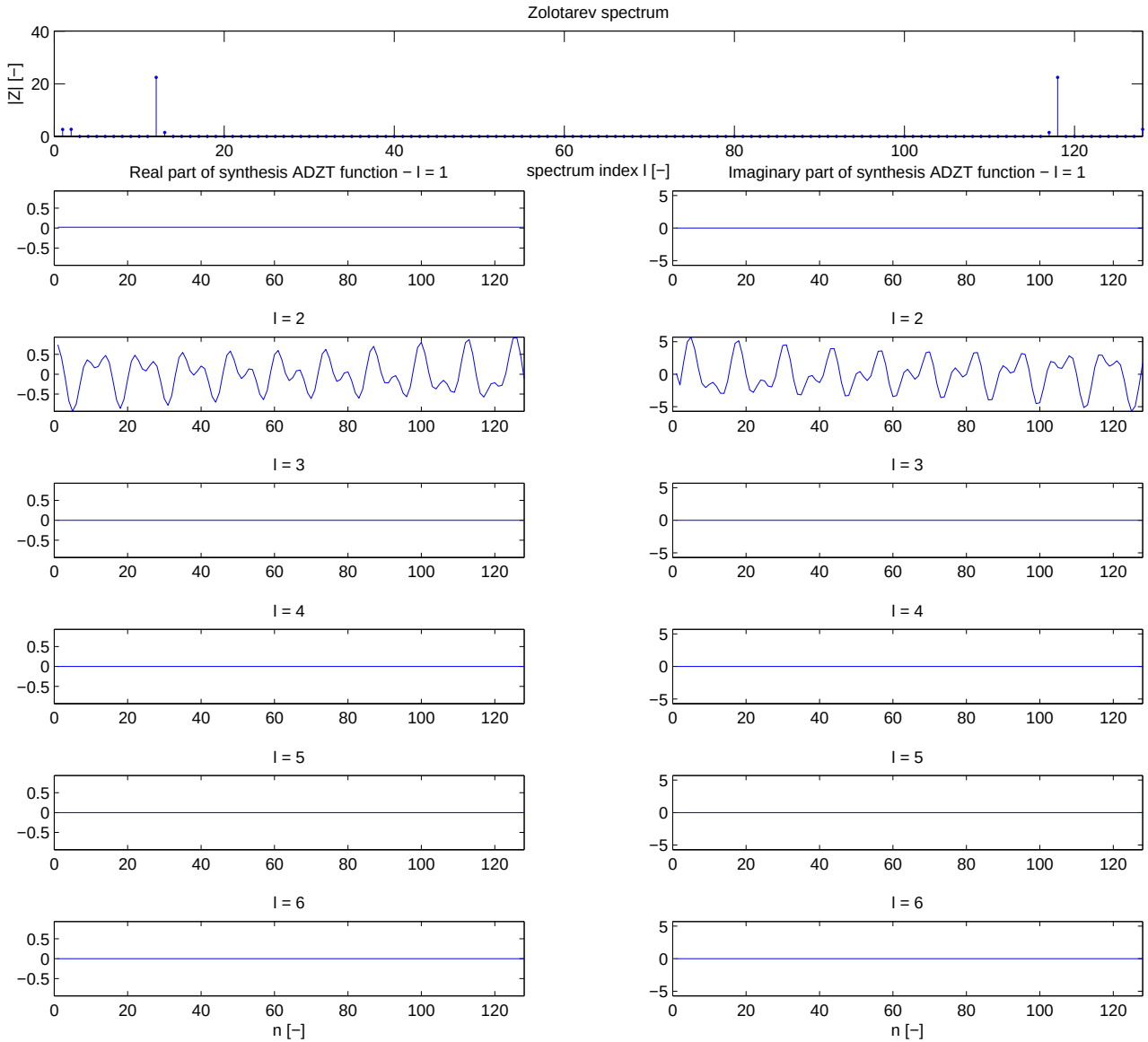
Analysis functions of DZT - part 1



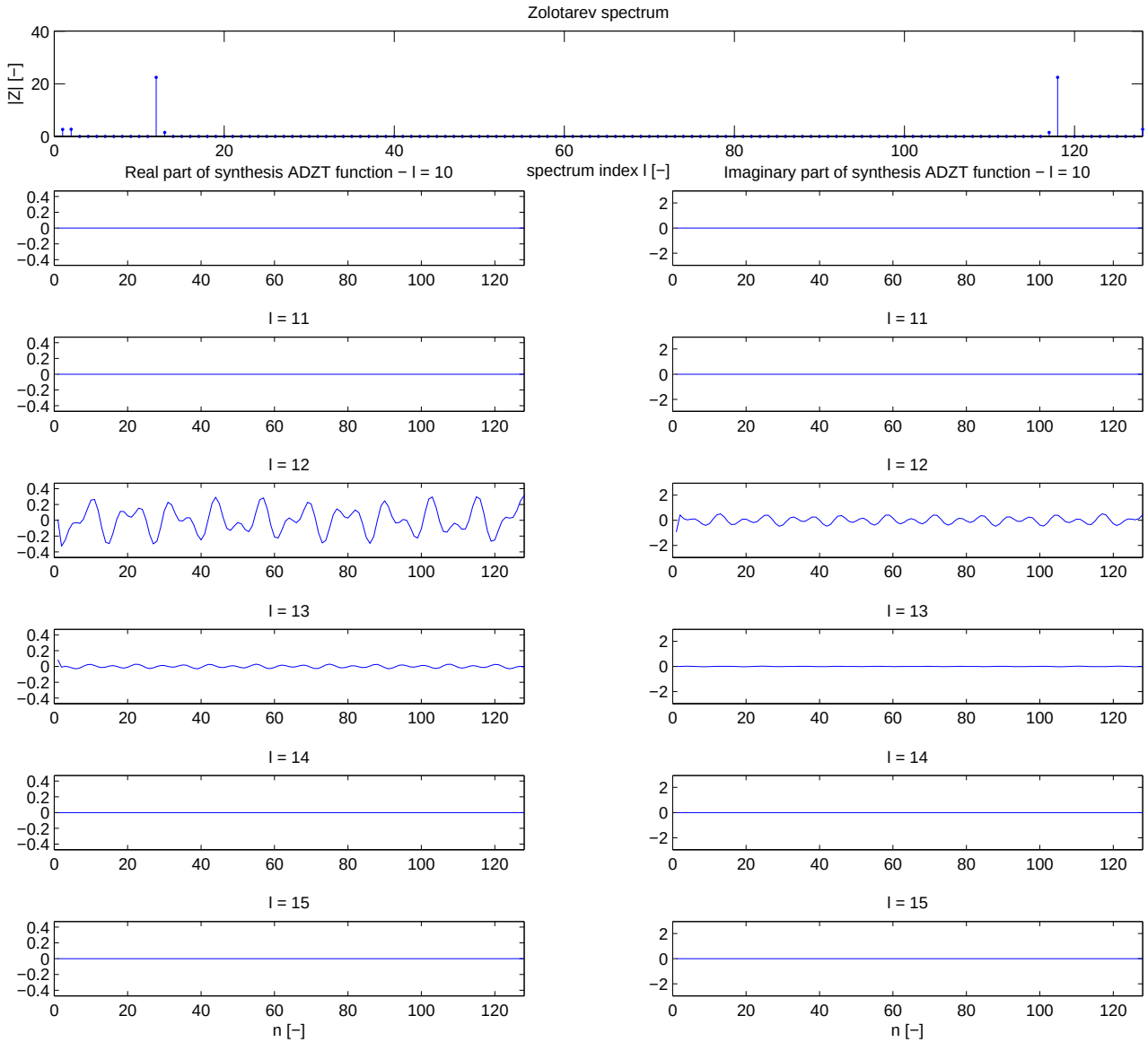
Synthesis functions of DZT - part 1



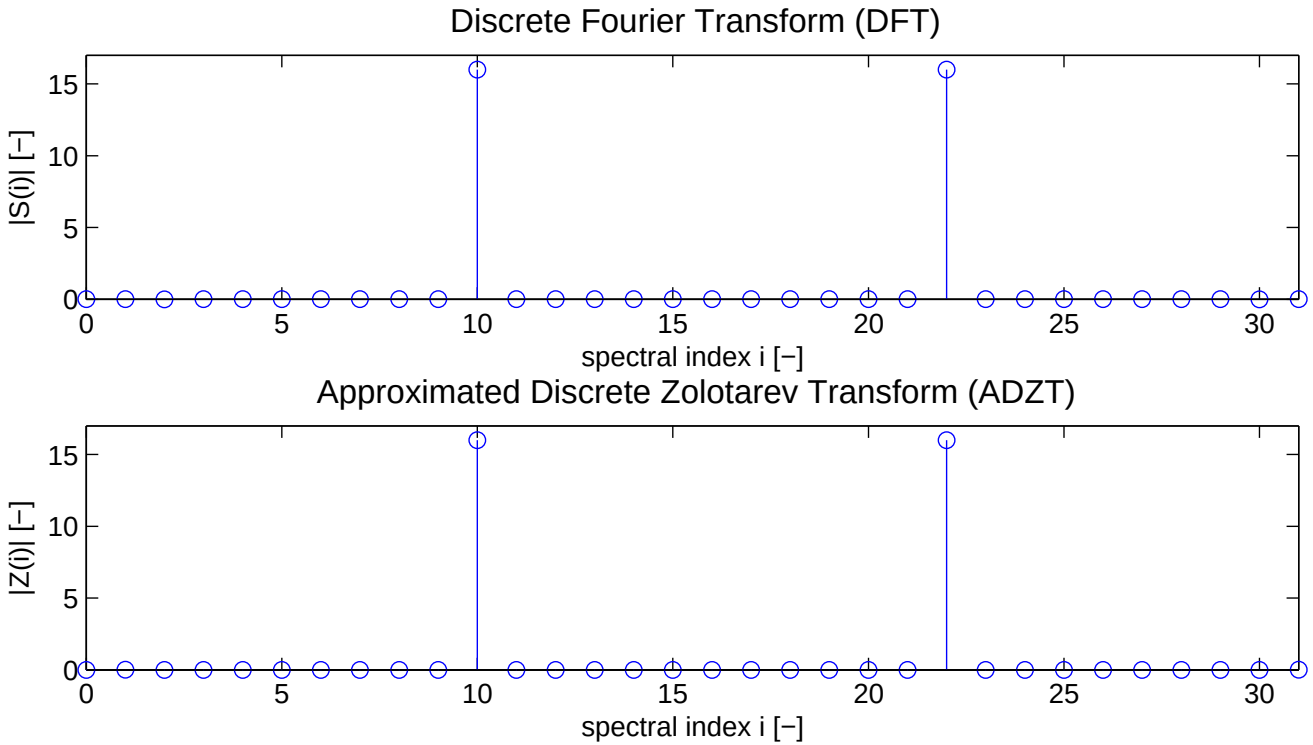
Analysis functions of DZT - part 2



Synthesis functions of DZT - part 2

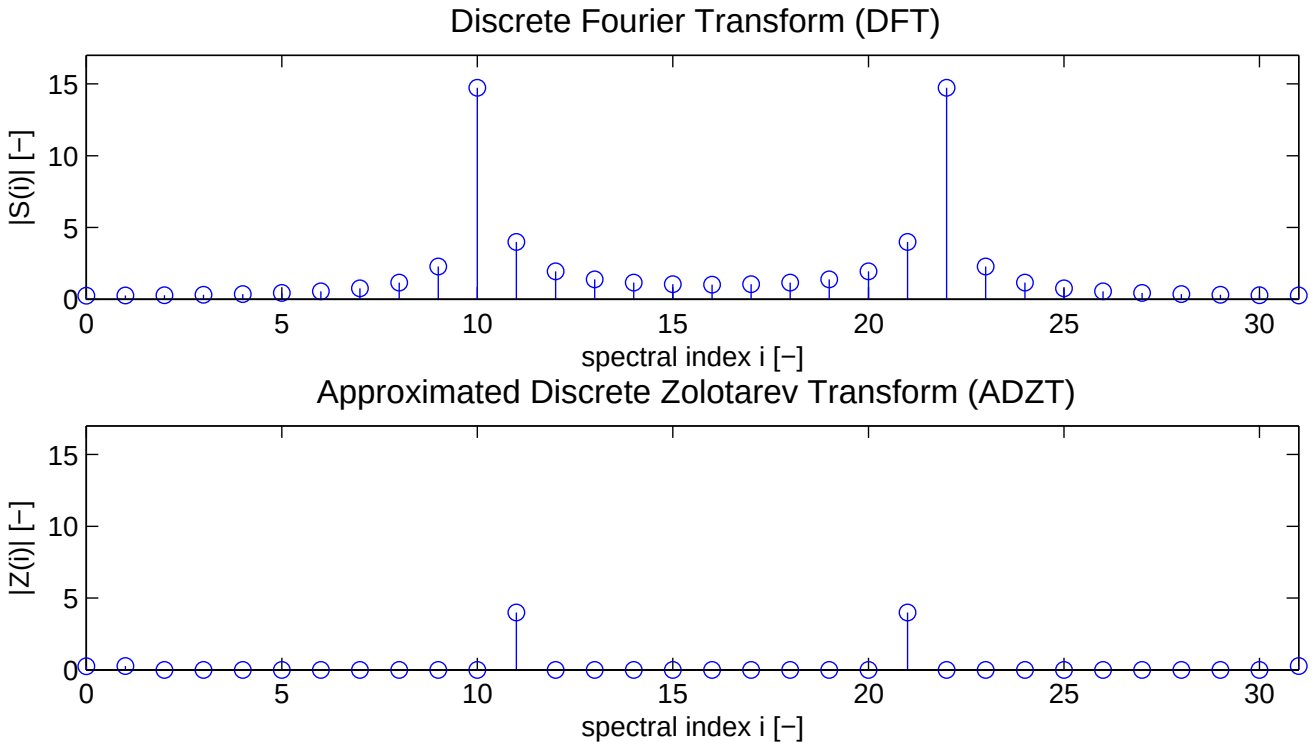


Suppression of spectral leakage by Approximated Discrete Zolotarev Transform (ADZT)



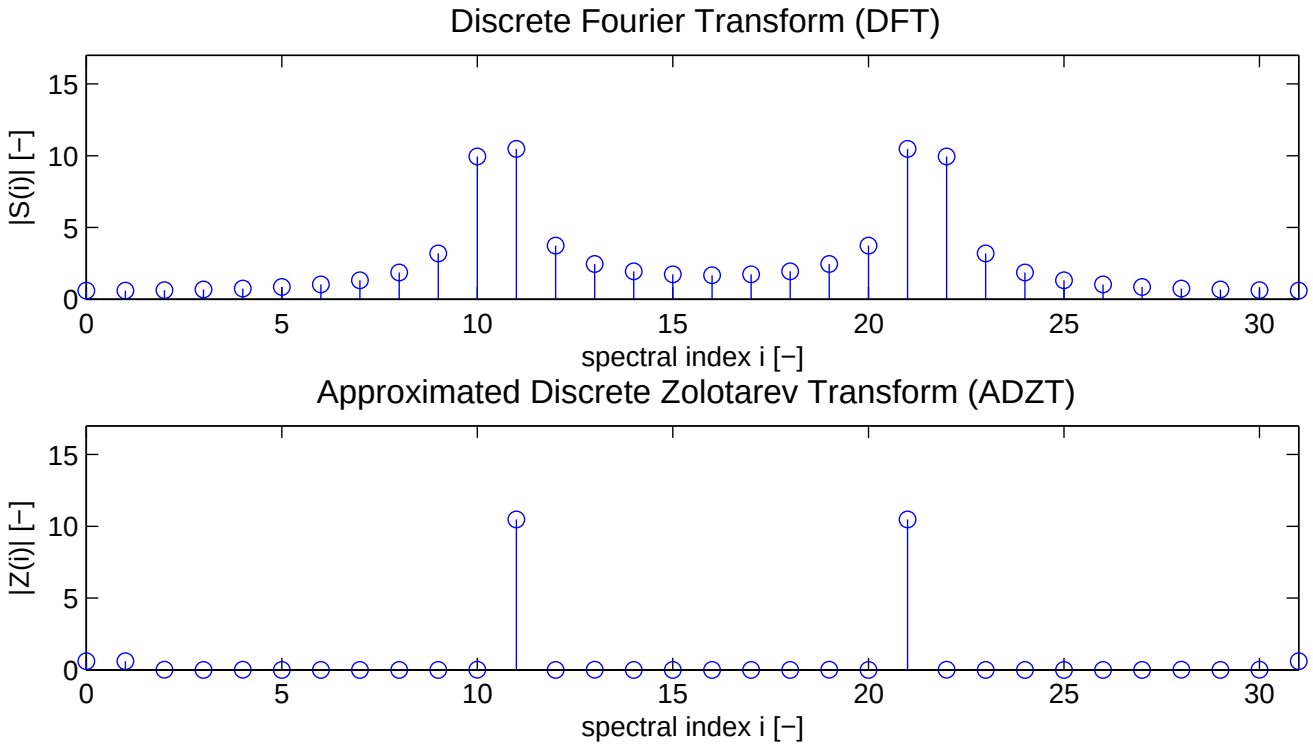
$$s(n) = \sin\left(10\frac{2\pi n}{N}\right), N = 32, n = 0, \ldots N-1$$

Suppression of spectral leakage by Approximated Discrete Zolotarev Transform (ADZT)



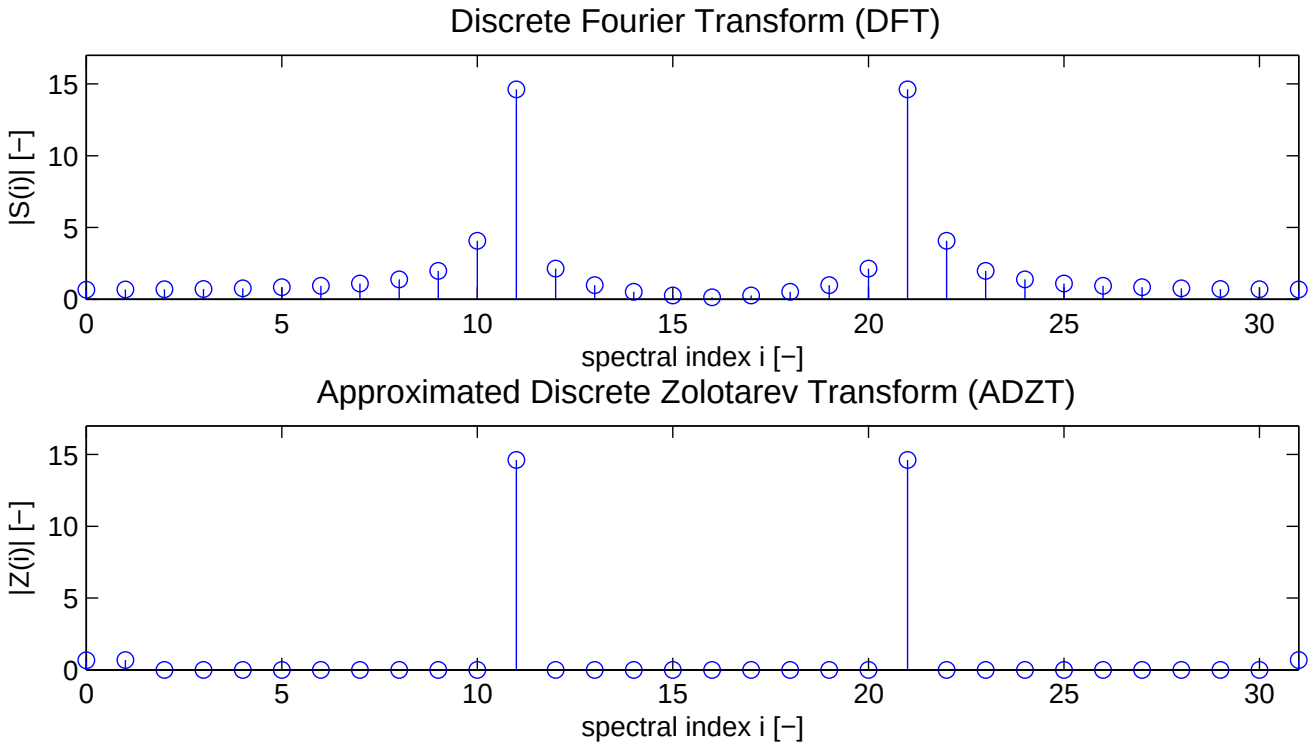
$$s(n) = \sin\left(10.2\frac{2\pi n}{N}\right), N = 32, n = 0, \ldots N-1$$

Suppression of spectral leakage by Approximated Discrete Zolotarev Transform (ADZT)



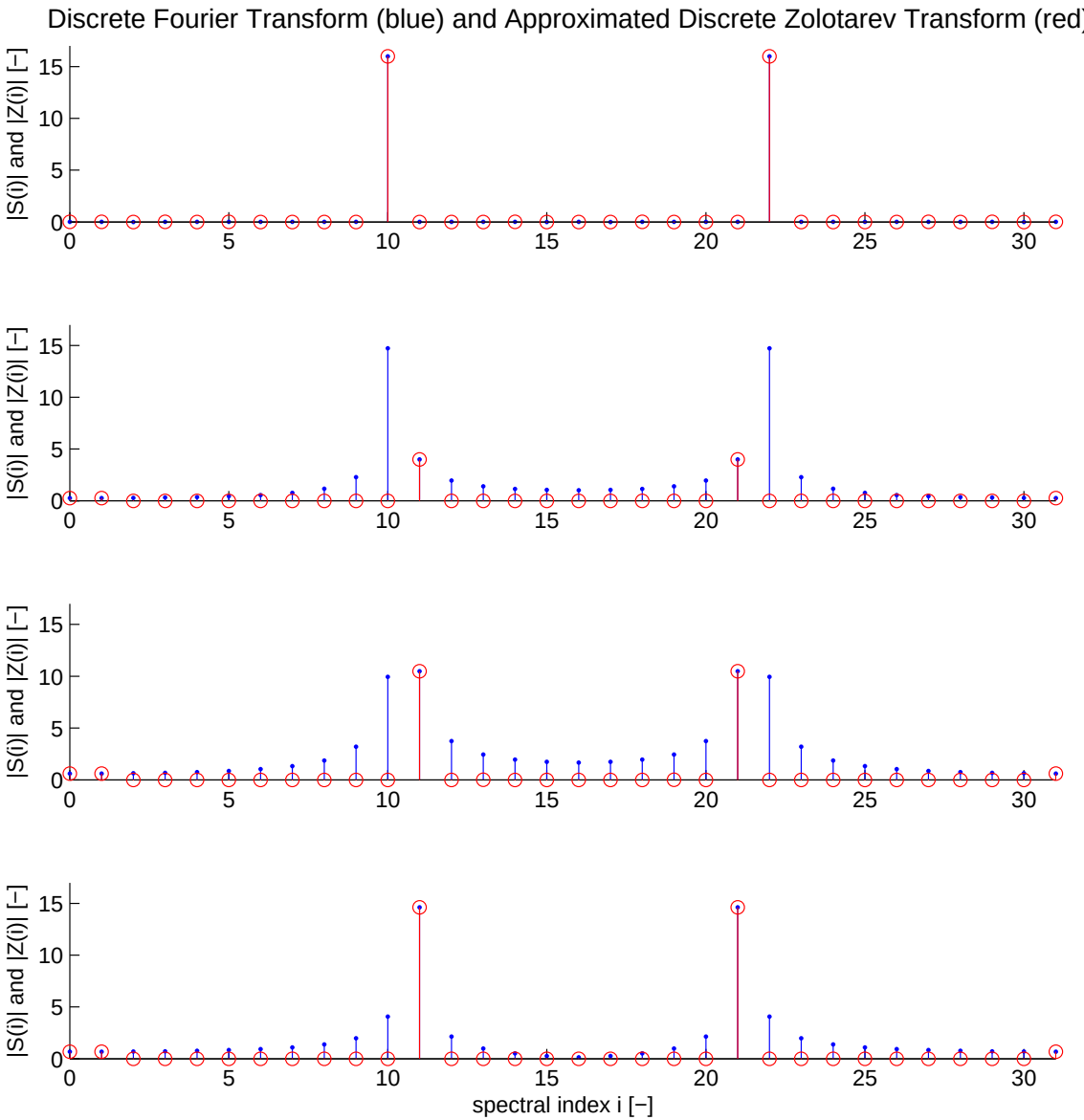
$$s(n) = \sin\left(10.3\frac{2\pi n}{N}\right), N = 32, n = 0, \ldots N-1$$

Suppression of spectral leakage by Approximated Discrete Zolotarev Transform (ADZT)

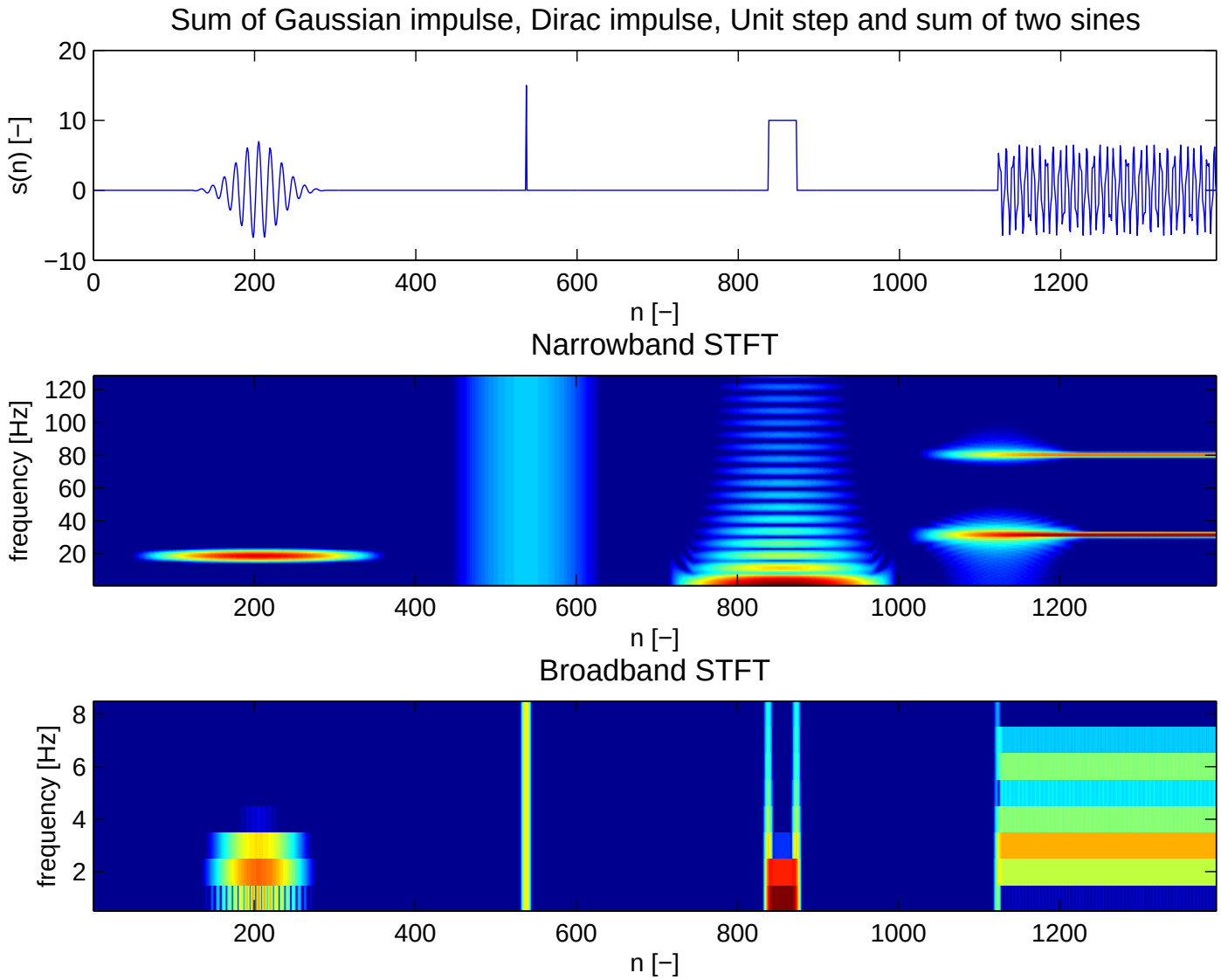


$$s(n) = \sin\left(10.8\frac{2\pi n}{N}\right), N = 32, n = 0, \ldots N-1$$

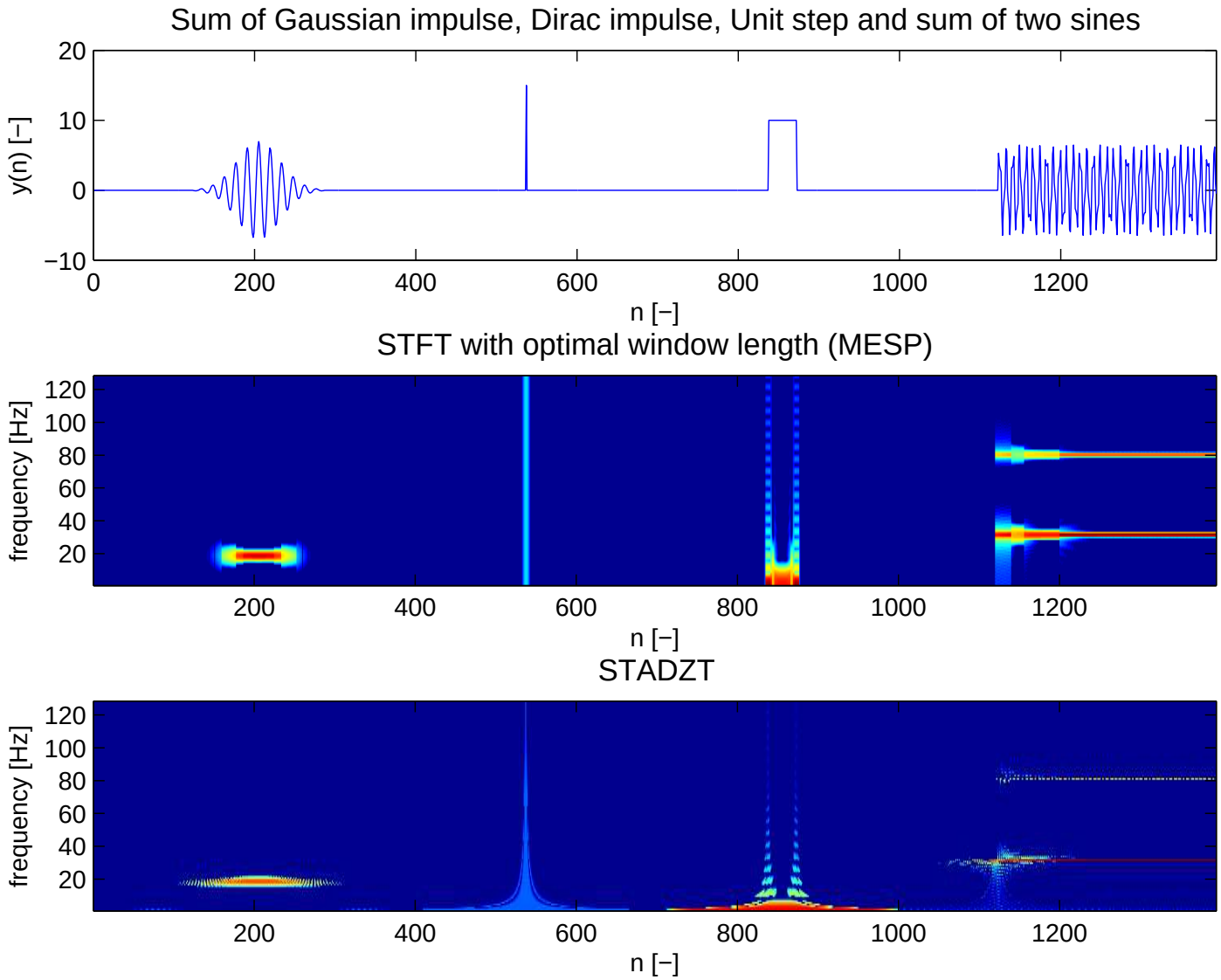
Suppression of spectral leakage by Approximated Discrete Zolotarev Transform (ADZT)



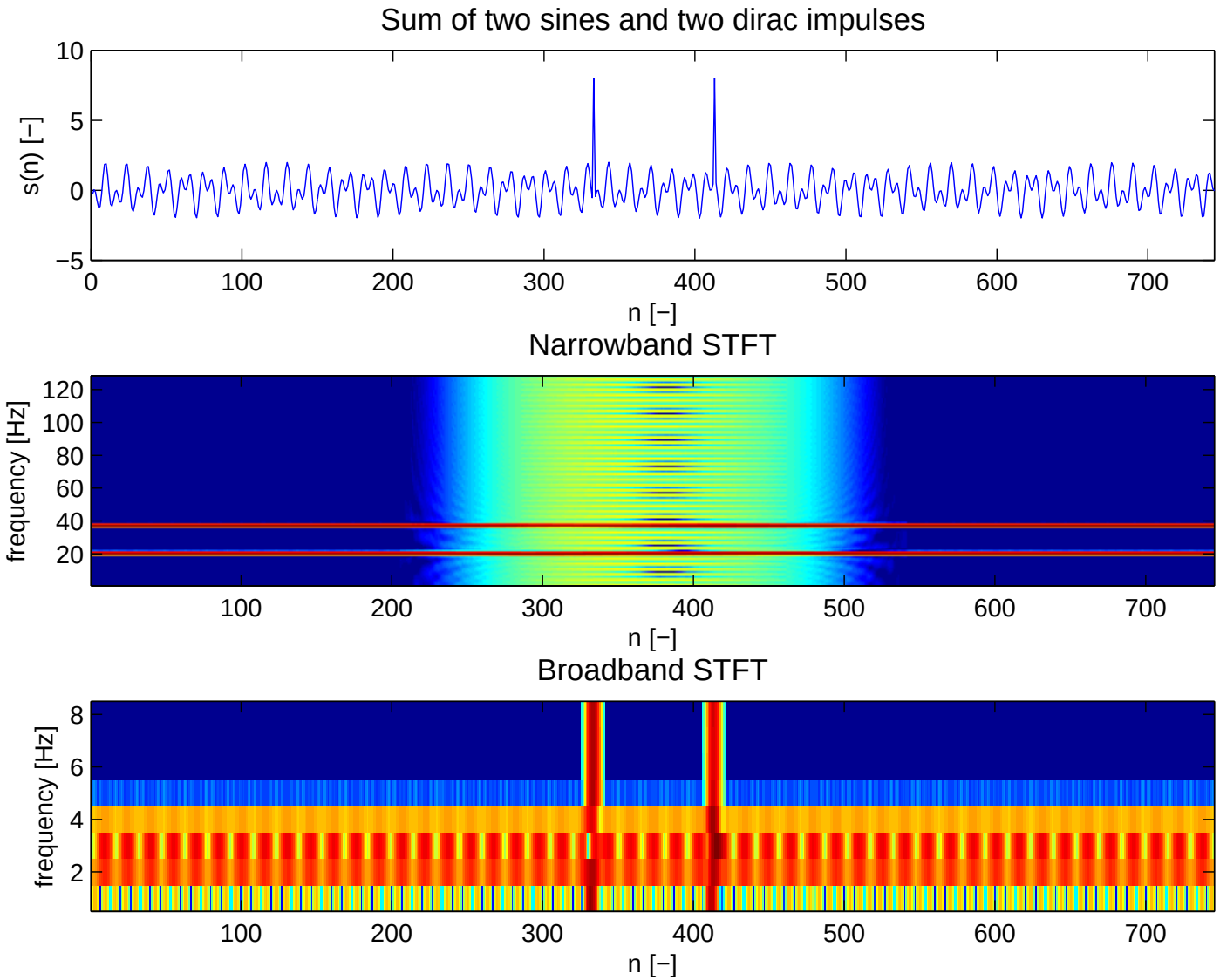
Narrowband and broadband STFT spectrograms of a test signal I



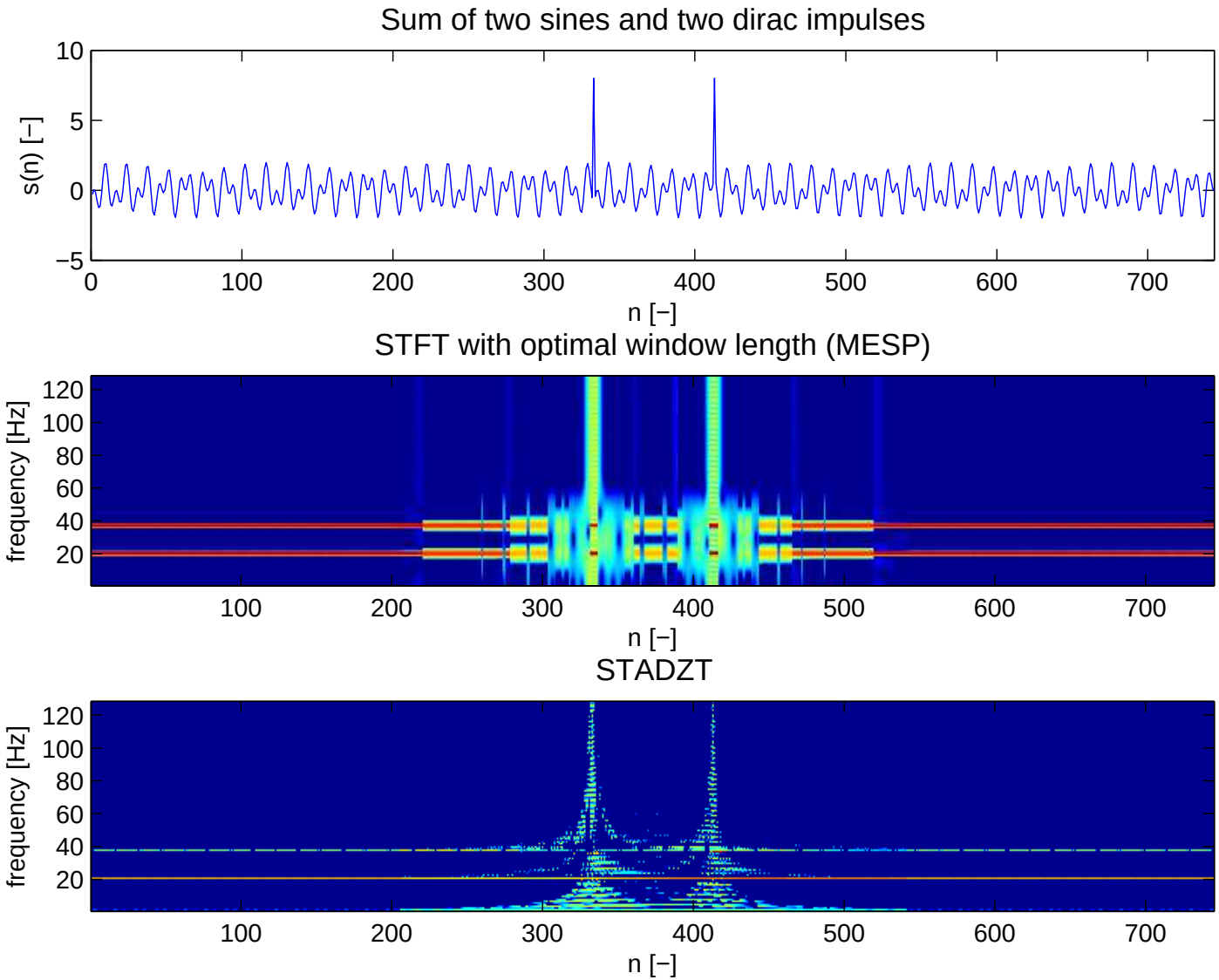
Adaptive STFT spectrogram and STADZT zologram



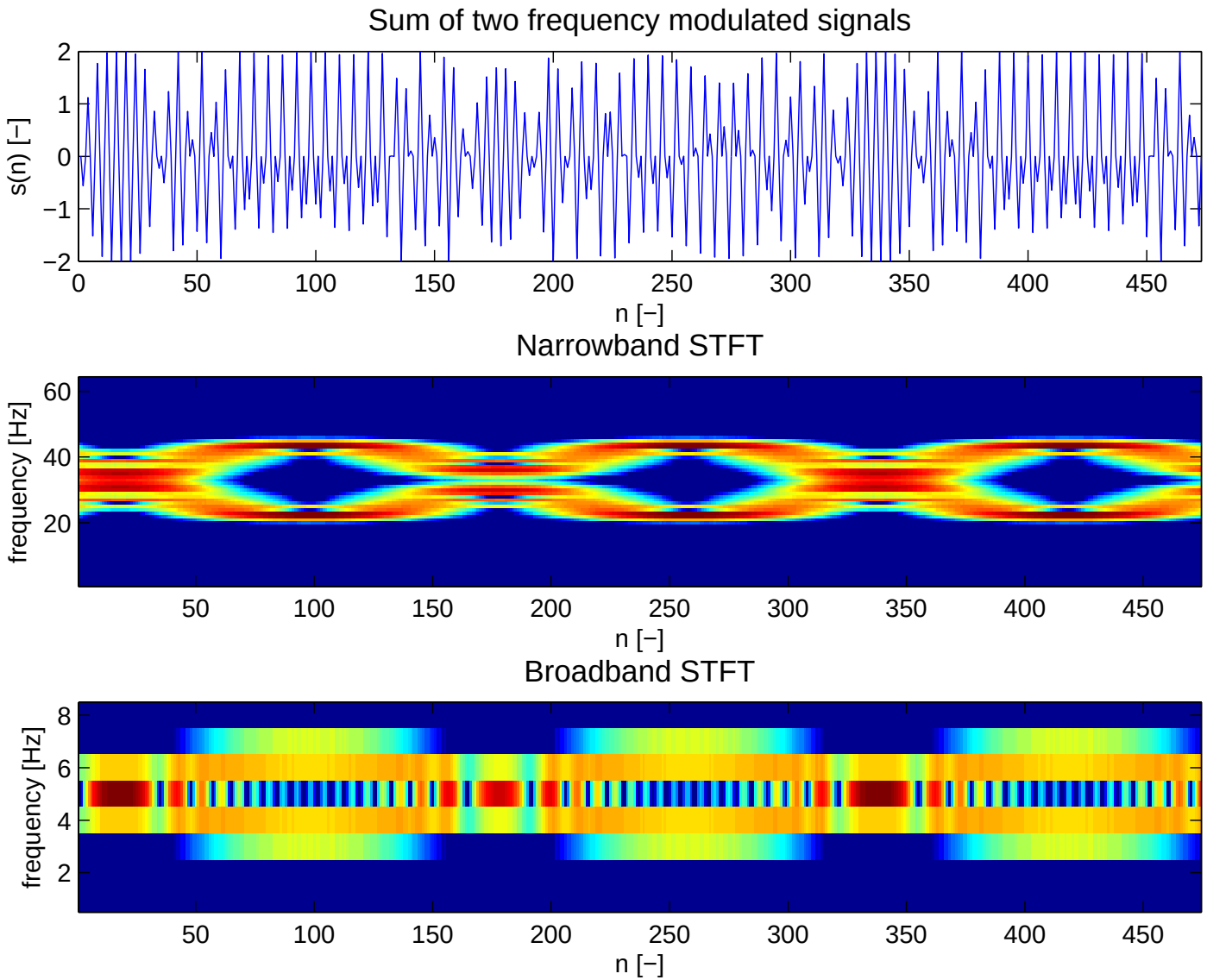
Narrowband and broadband STFT spectrograms of a test signal II



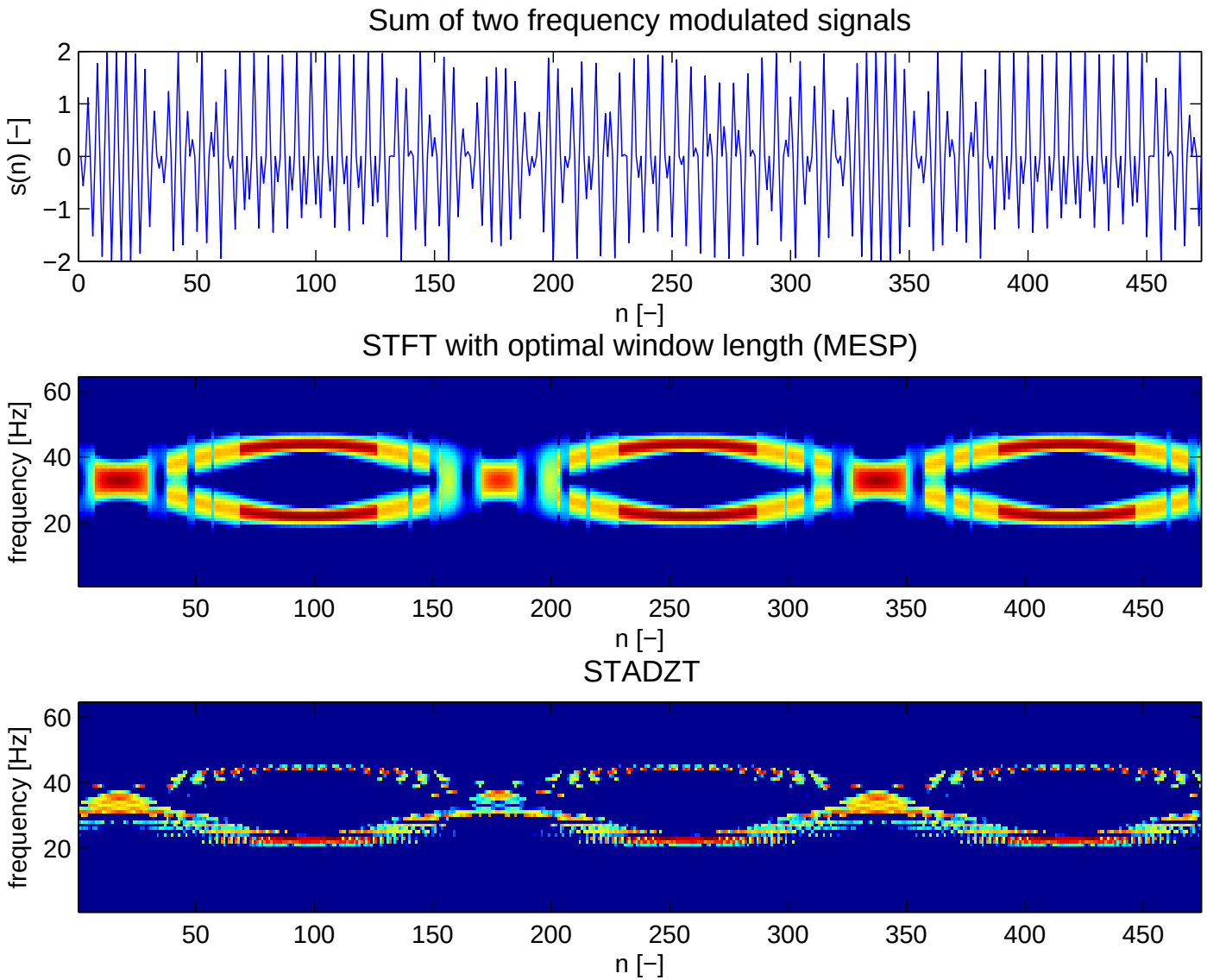
Adaptive STFT spectrogram and STADZT zologram



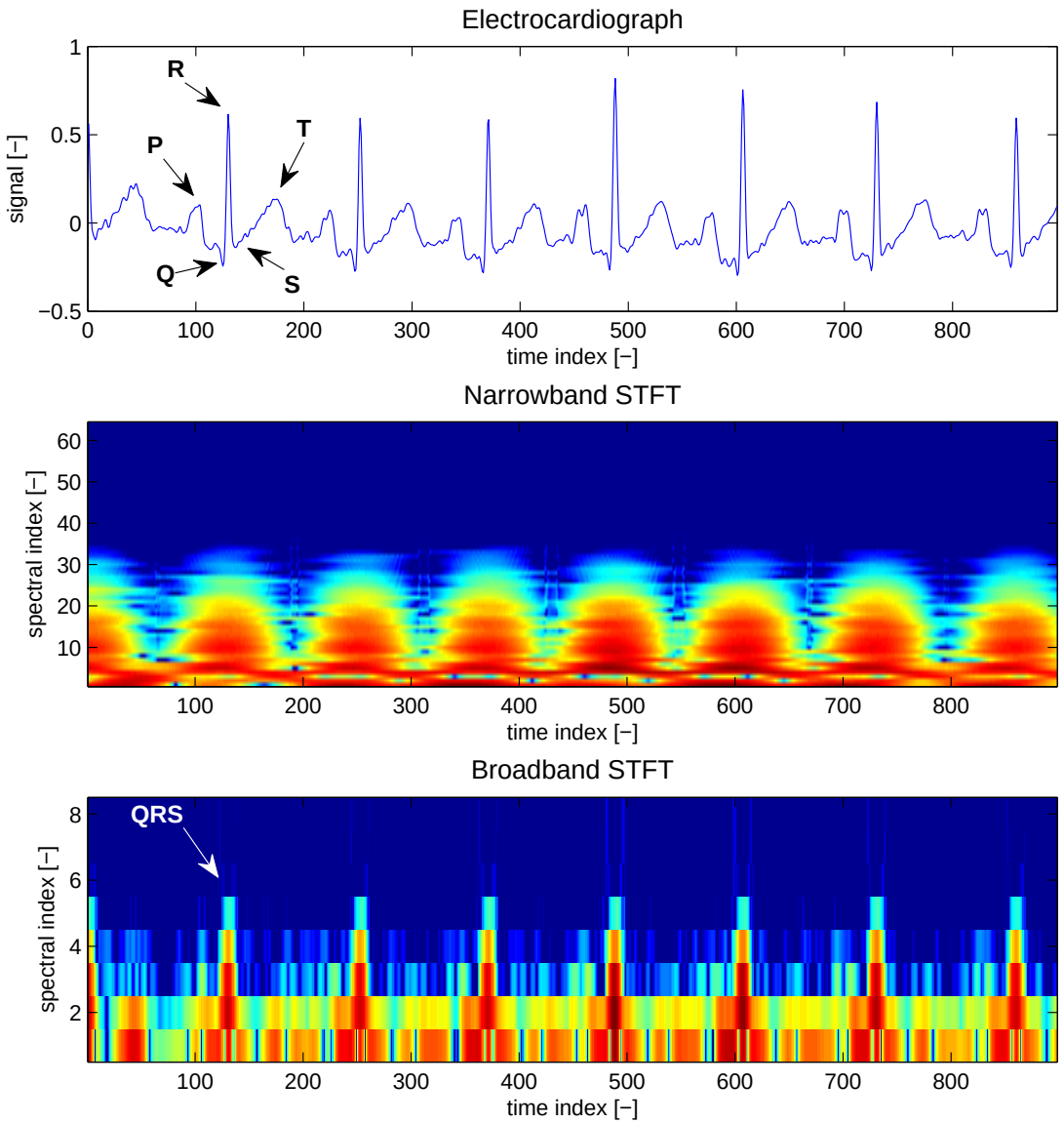
Narrowband and broadband STFT spectrograms of a test signal III



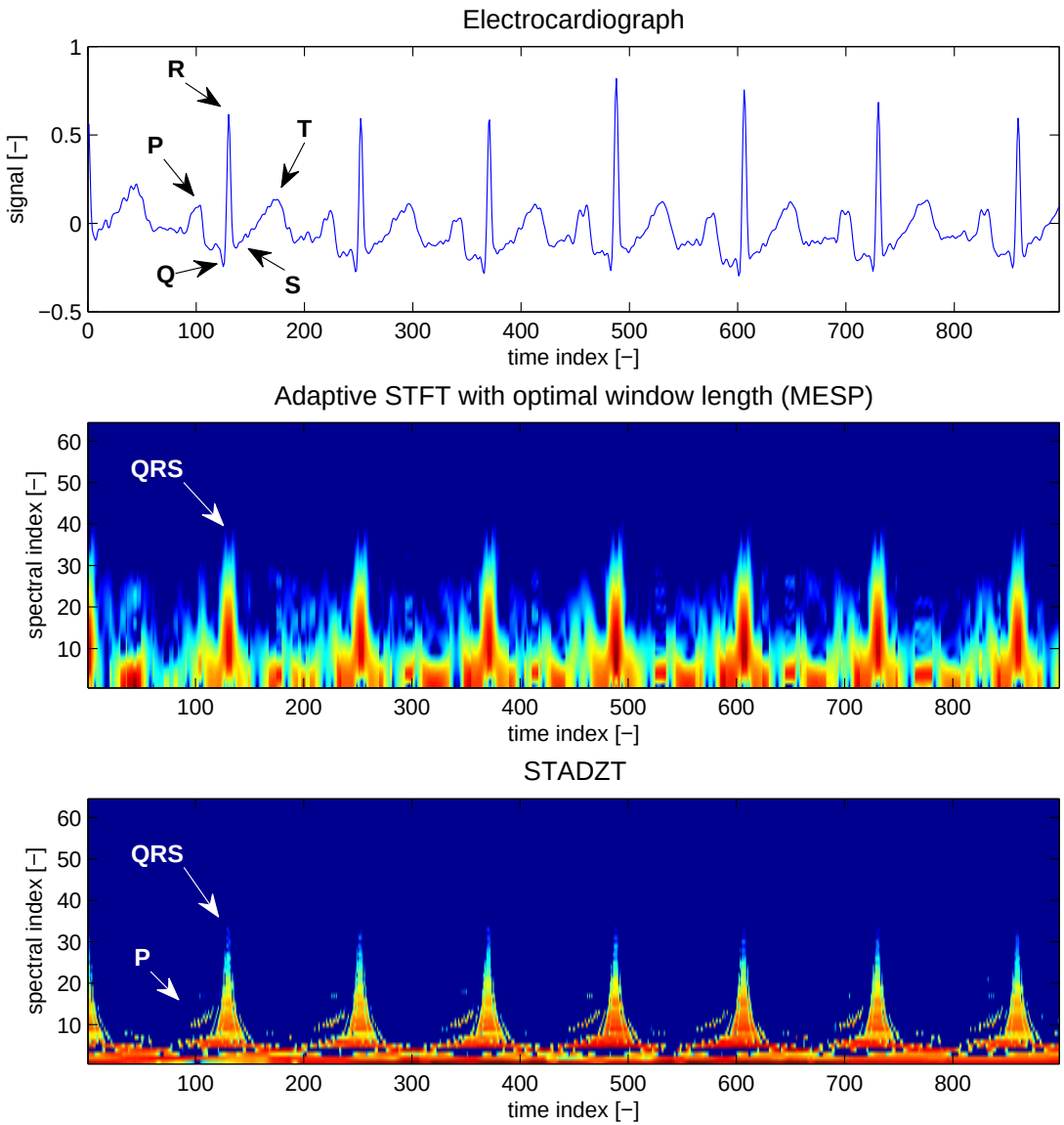
Adaptive STFT spectrogram and STADZT zologram



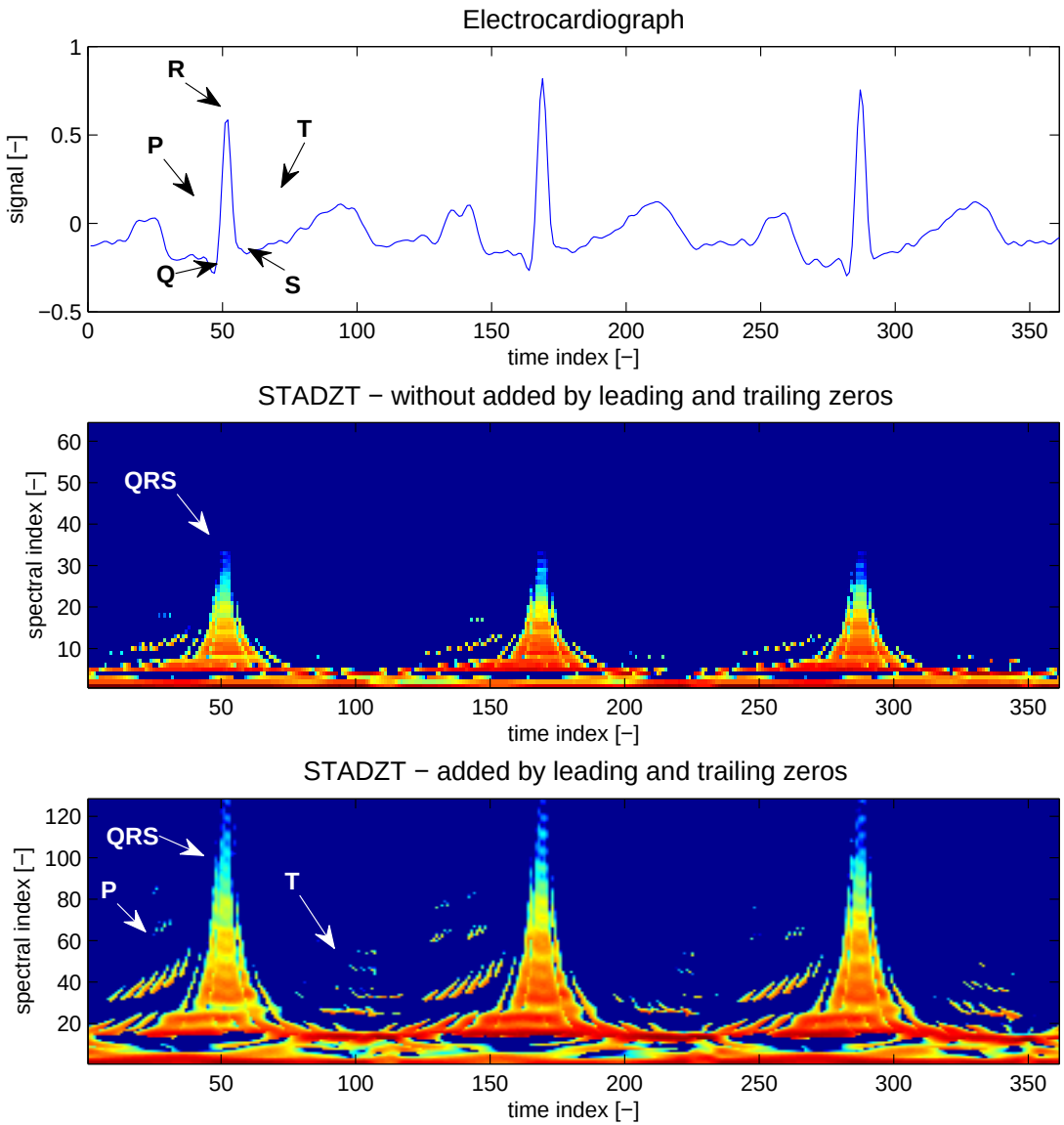
Narrowband and broadband STFT spectrograms of an ECG signal



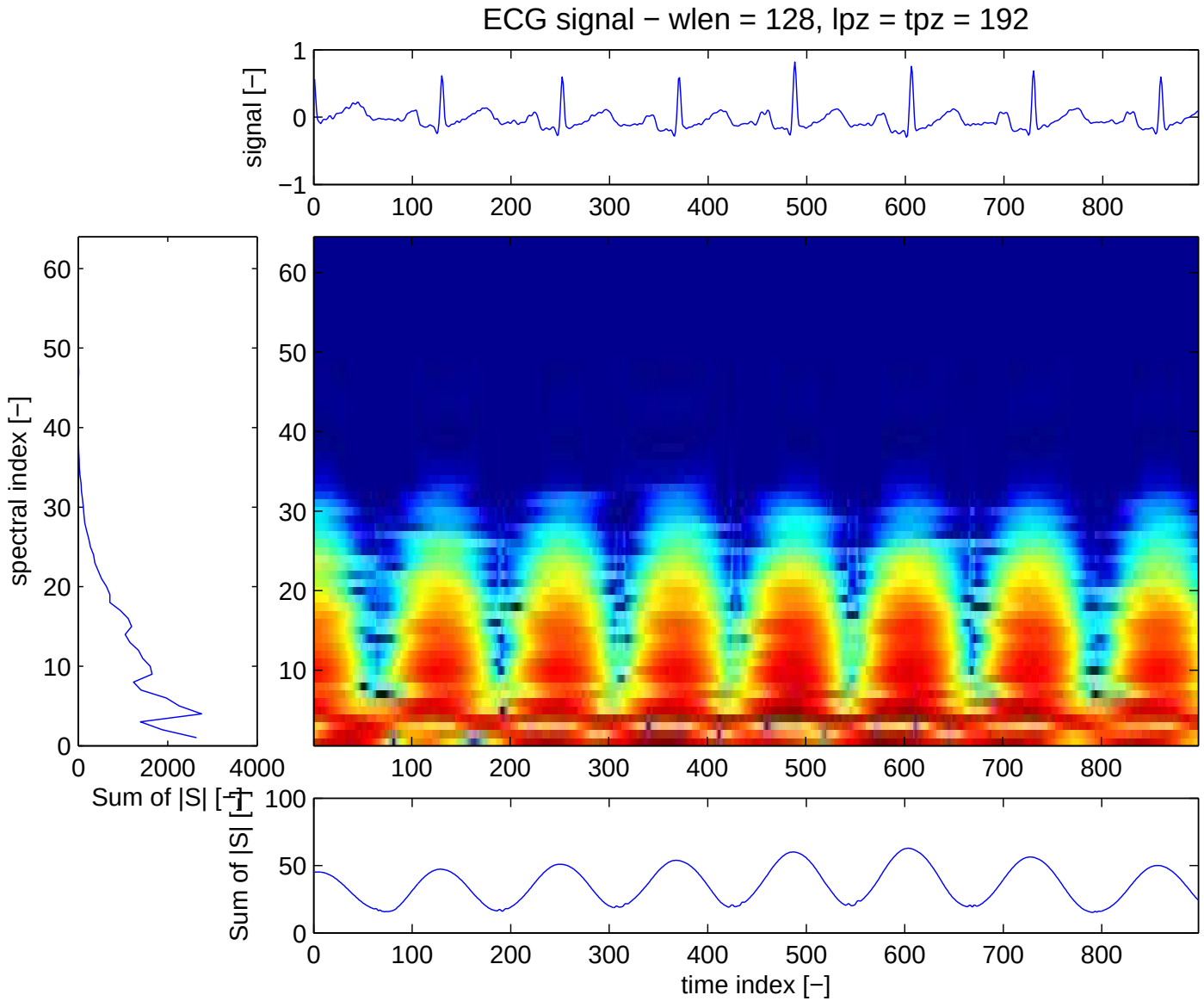
Adaptive STFT spectrogram and STADZT zologram



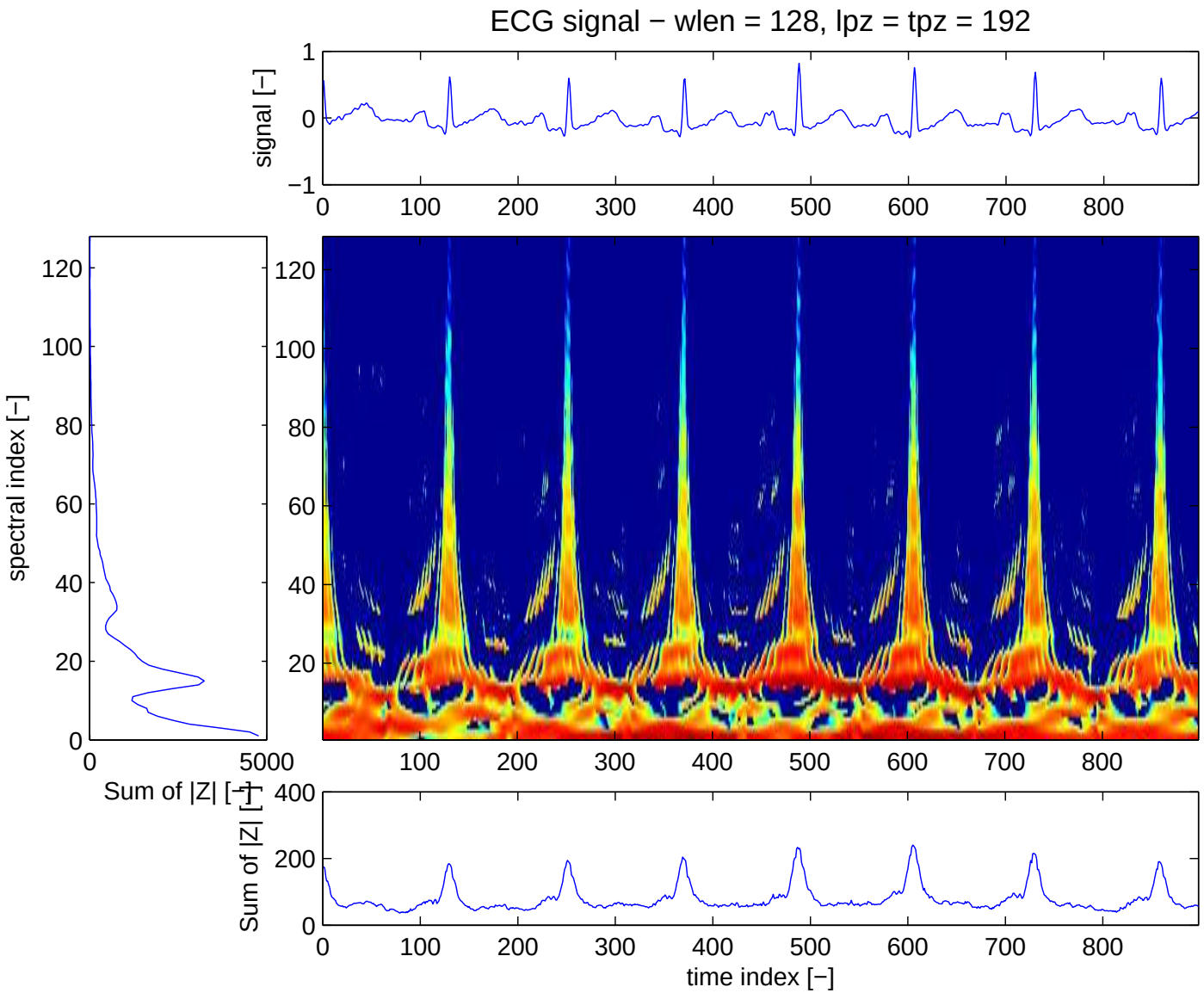
STADZT zologram with added leading and trailing zeros



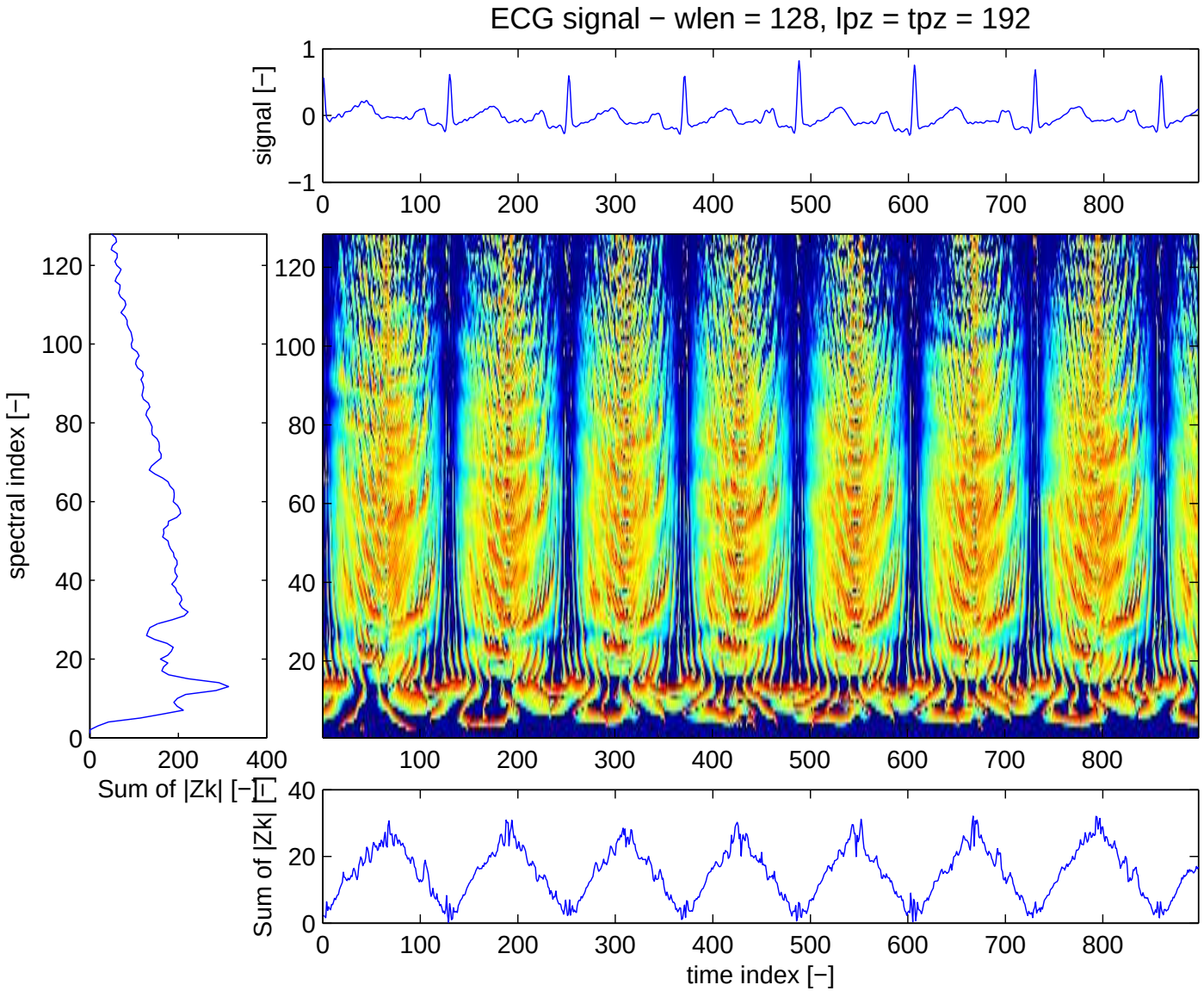
ECG signal - STFT spectrogram



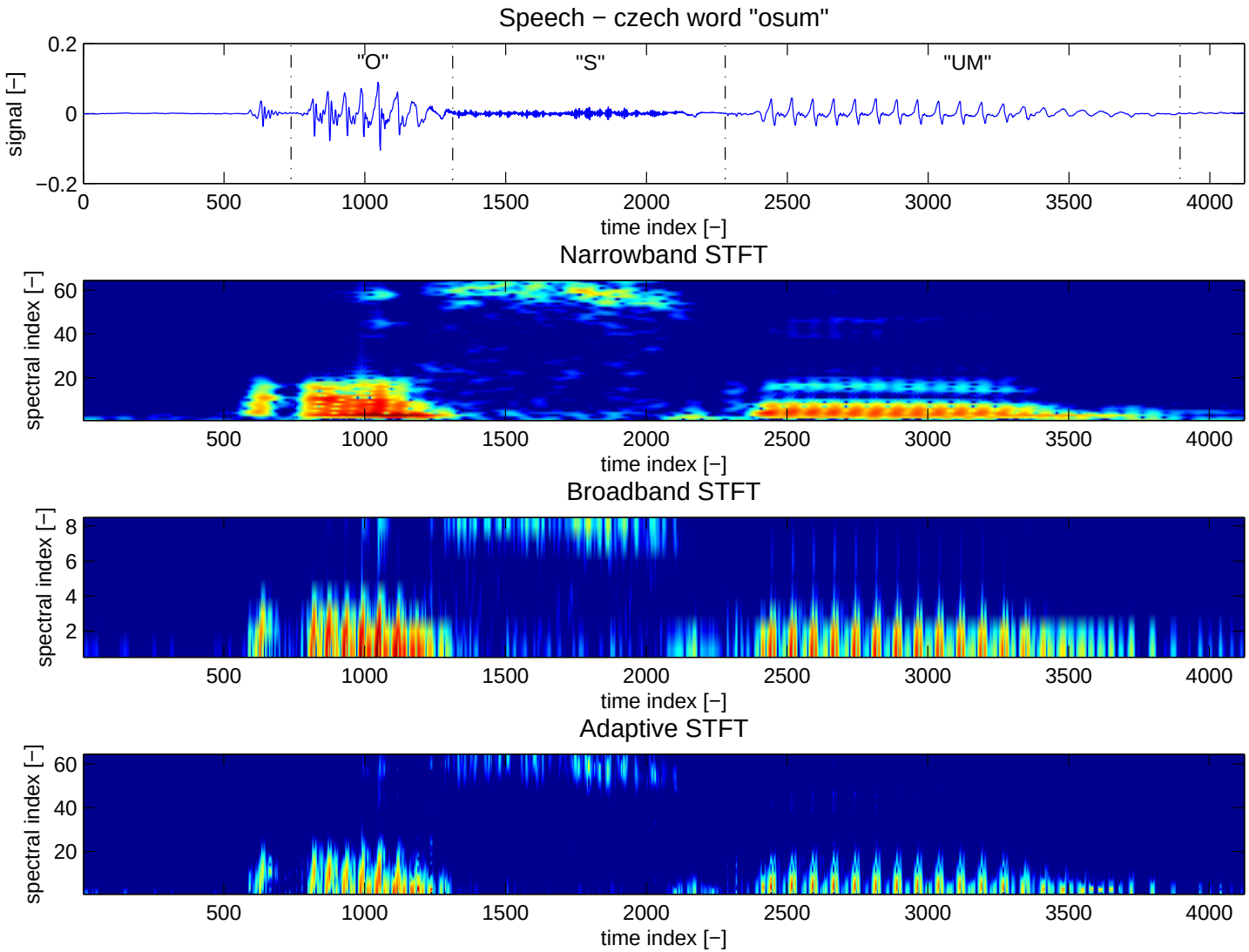
ECG signal - STADZT zologram with added leading and trailing zeros



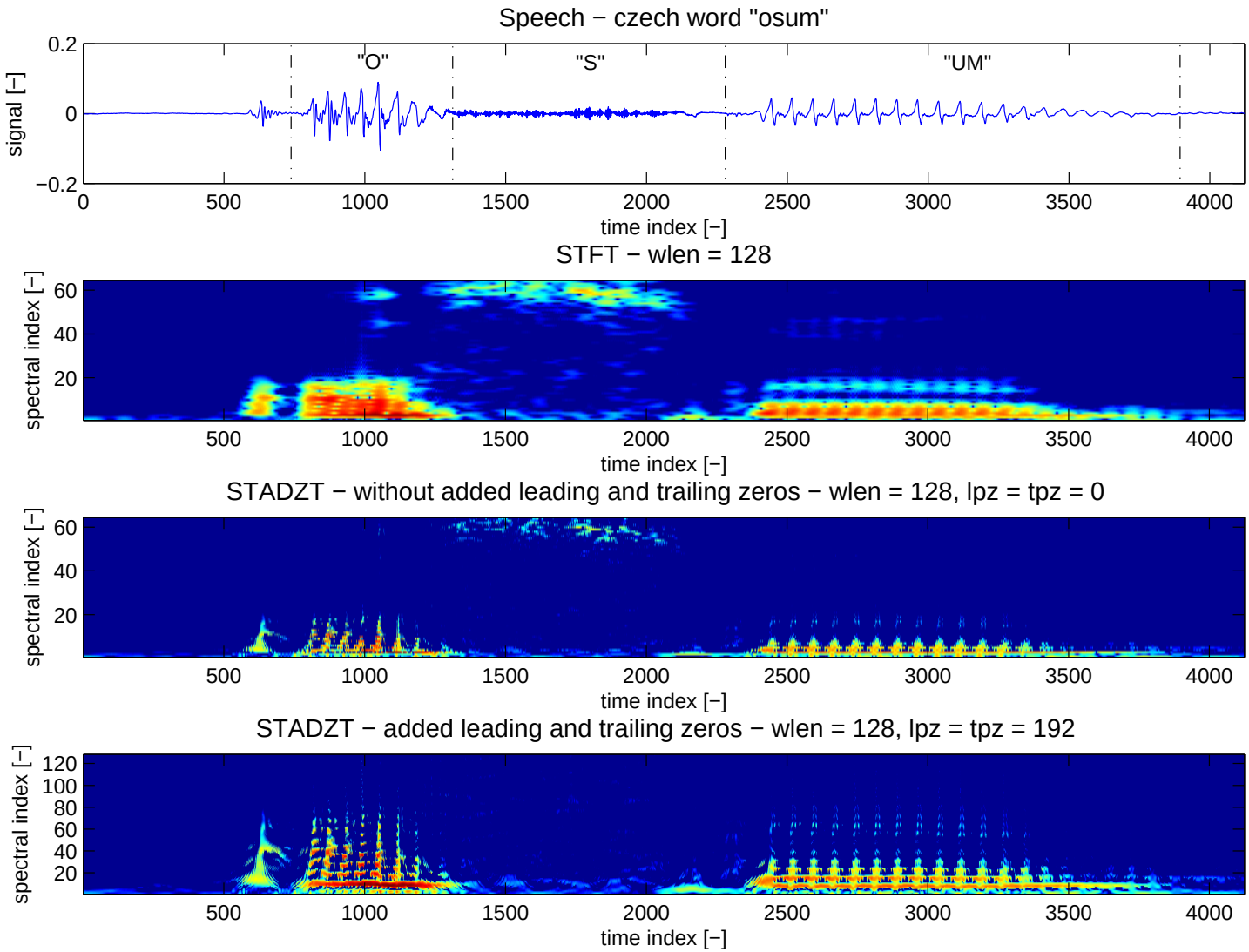
ECG signal - STADZT k-gram with added leading and trailing zeros



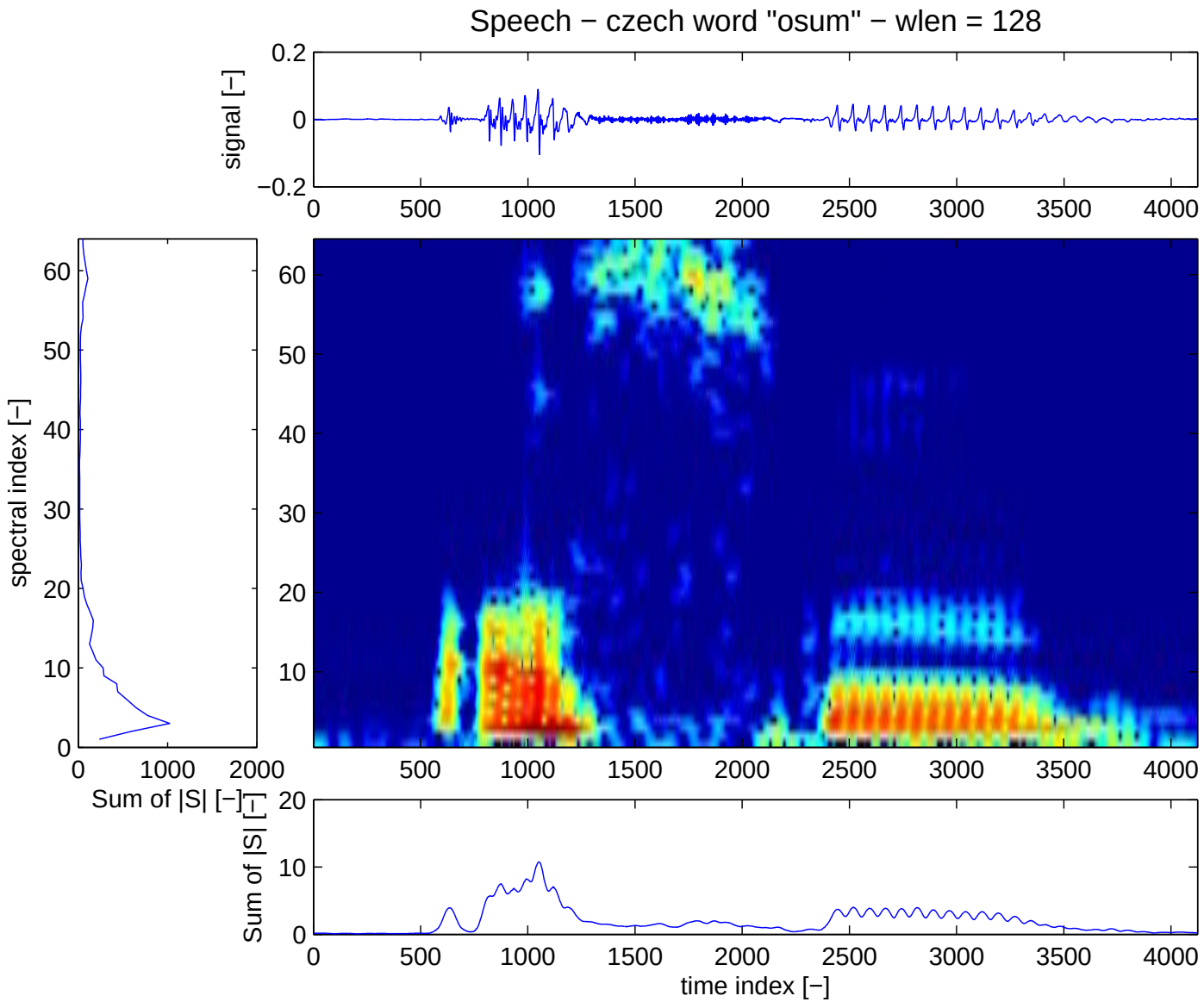
Broadband and narrowband STFT spectrograms and adaptive STFT spectrogram



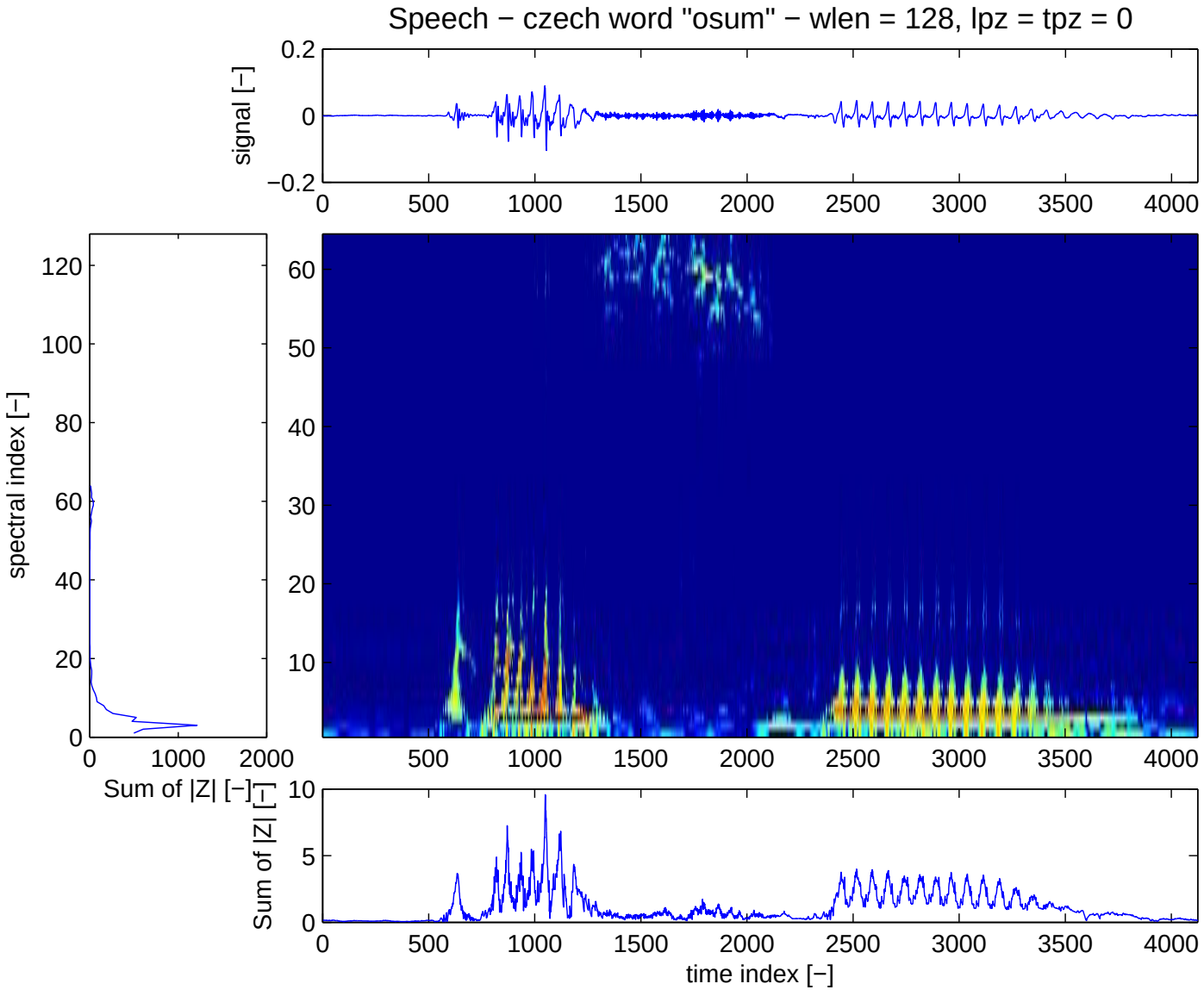
STFT spectrogram and STADZT zolograms with and without added leading and trailing zeros



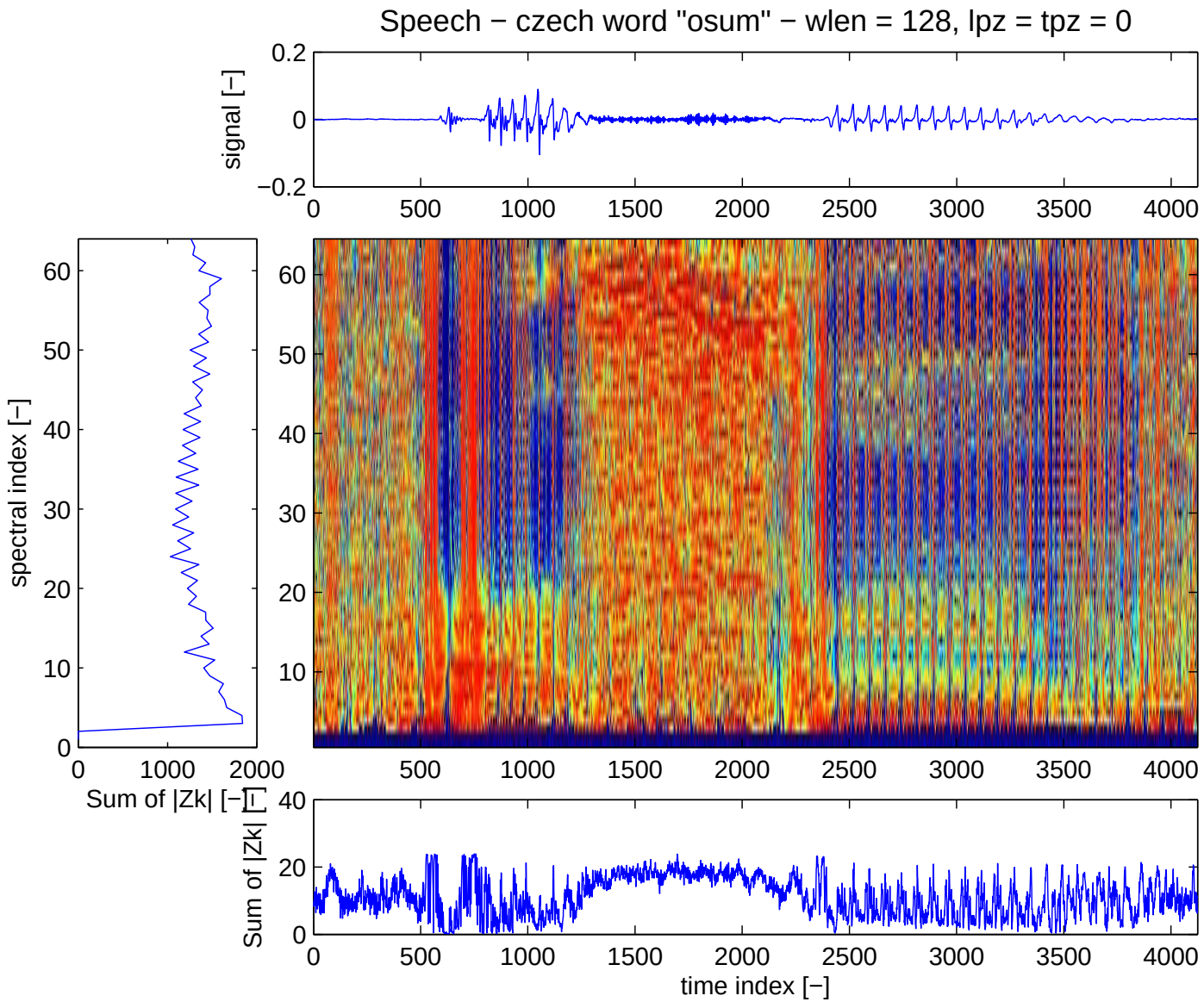
Speech - STFT spectrogram



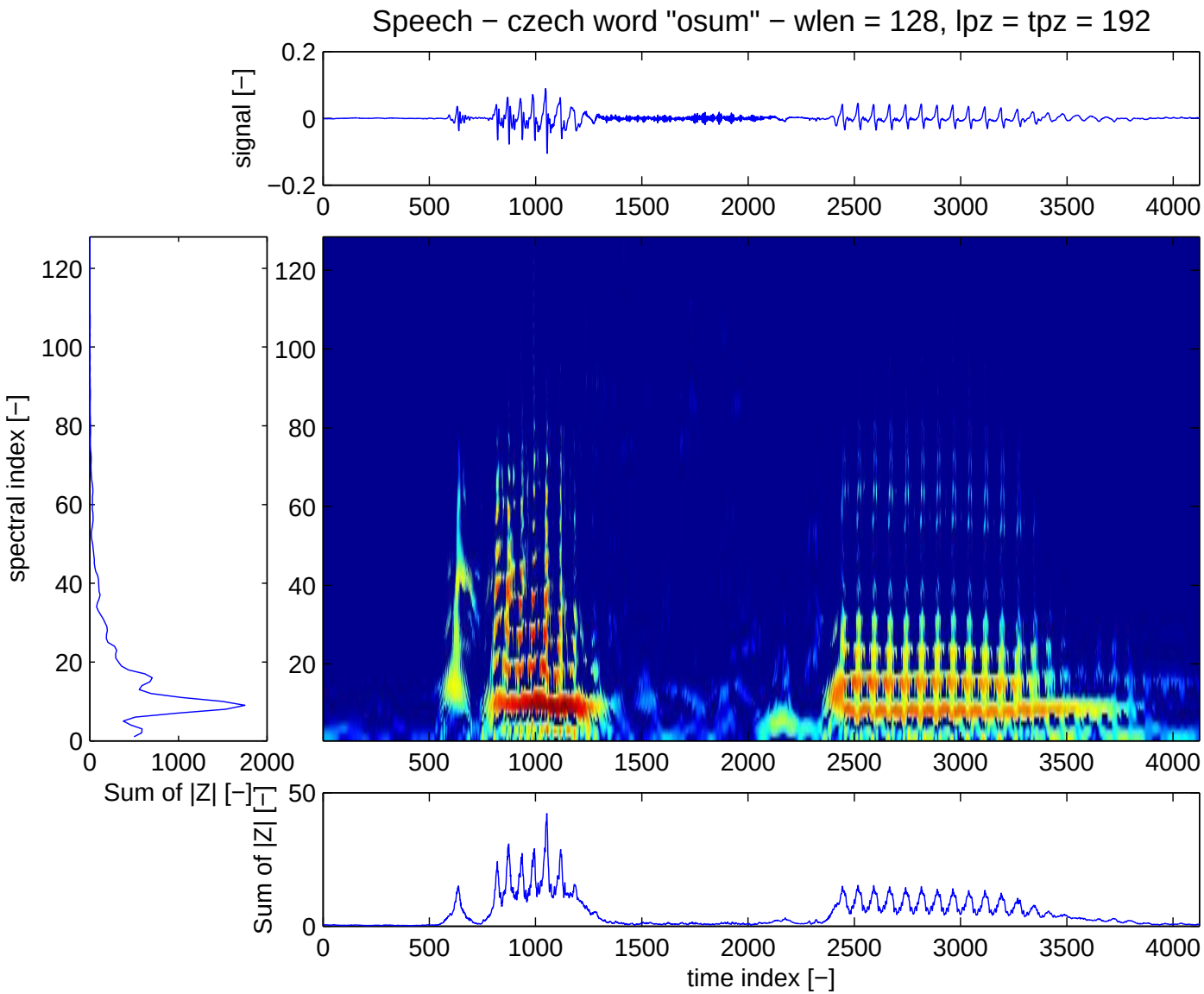
Speech - STADZT zologram



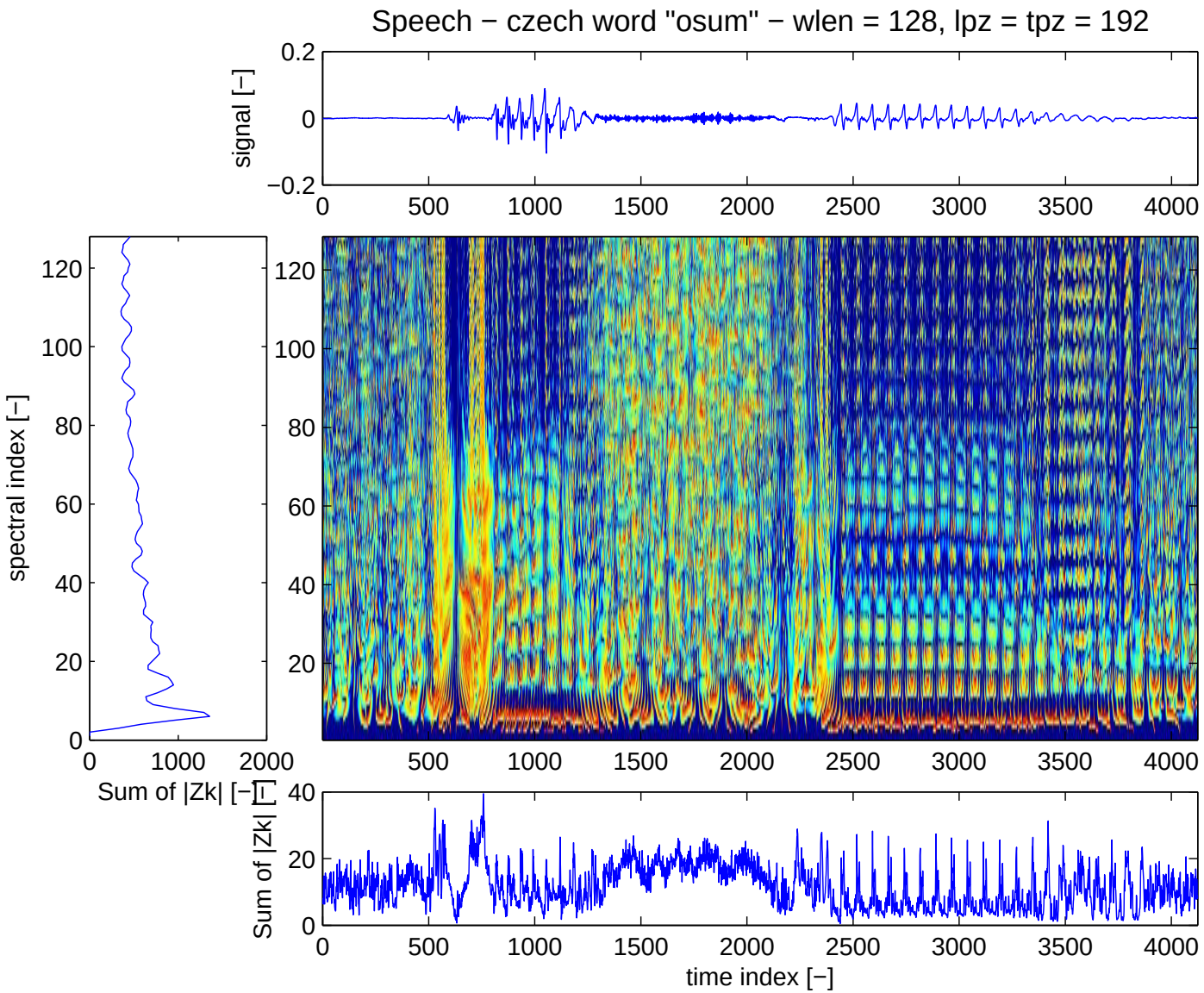
Speech - STADZT k-gram



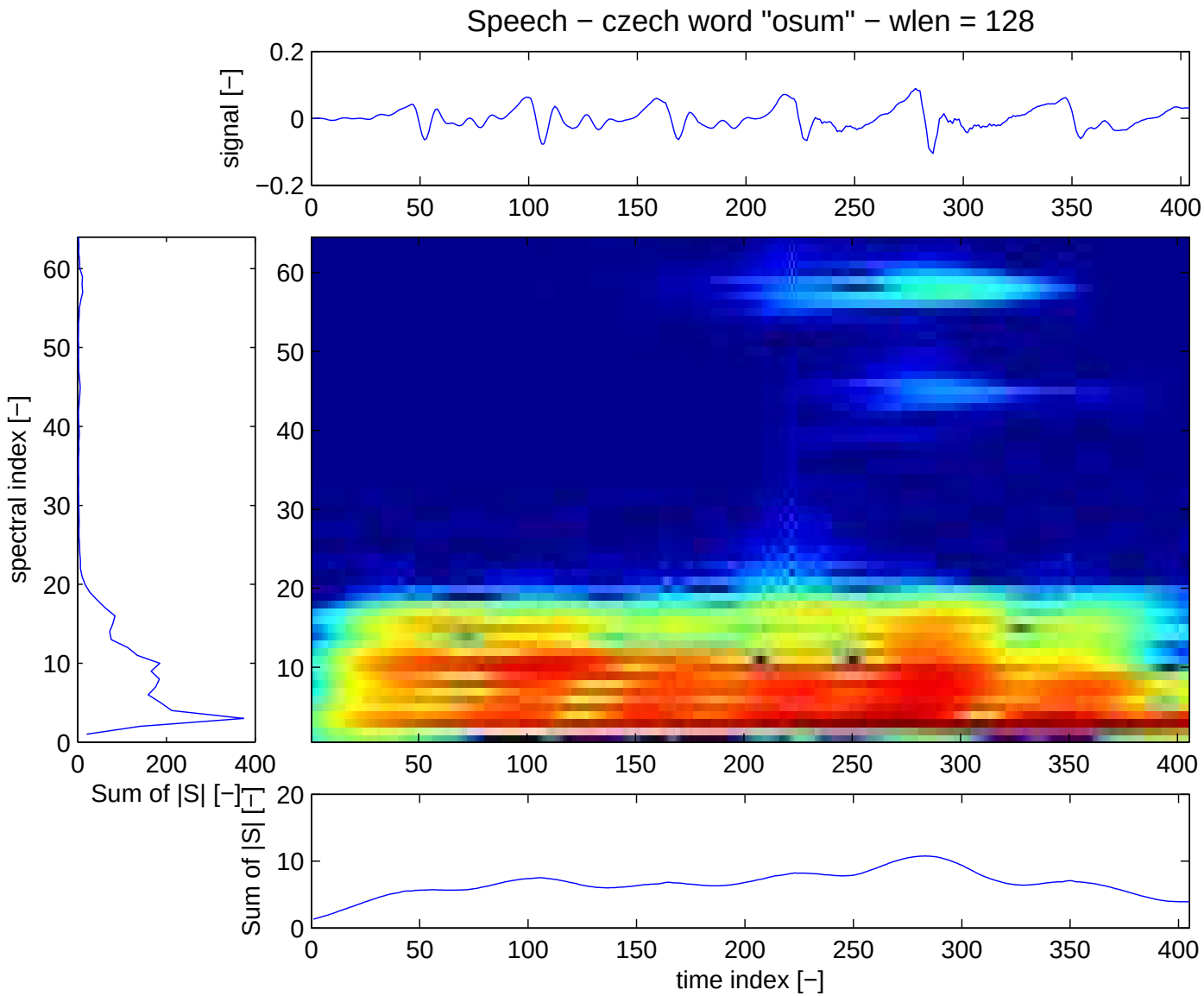
Speech - STADZT zologram with added leading and trailing zeros



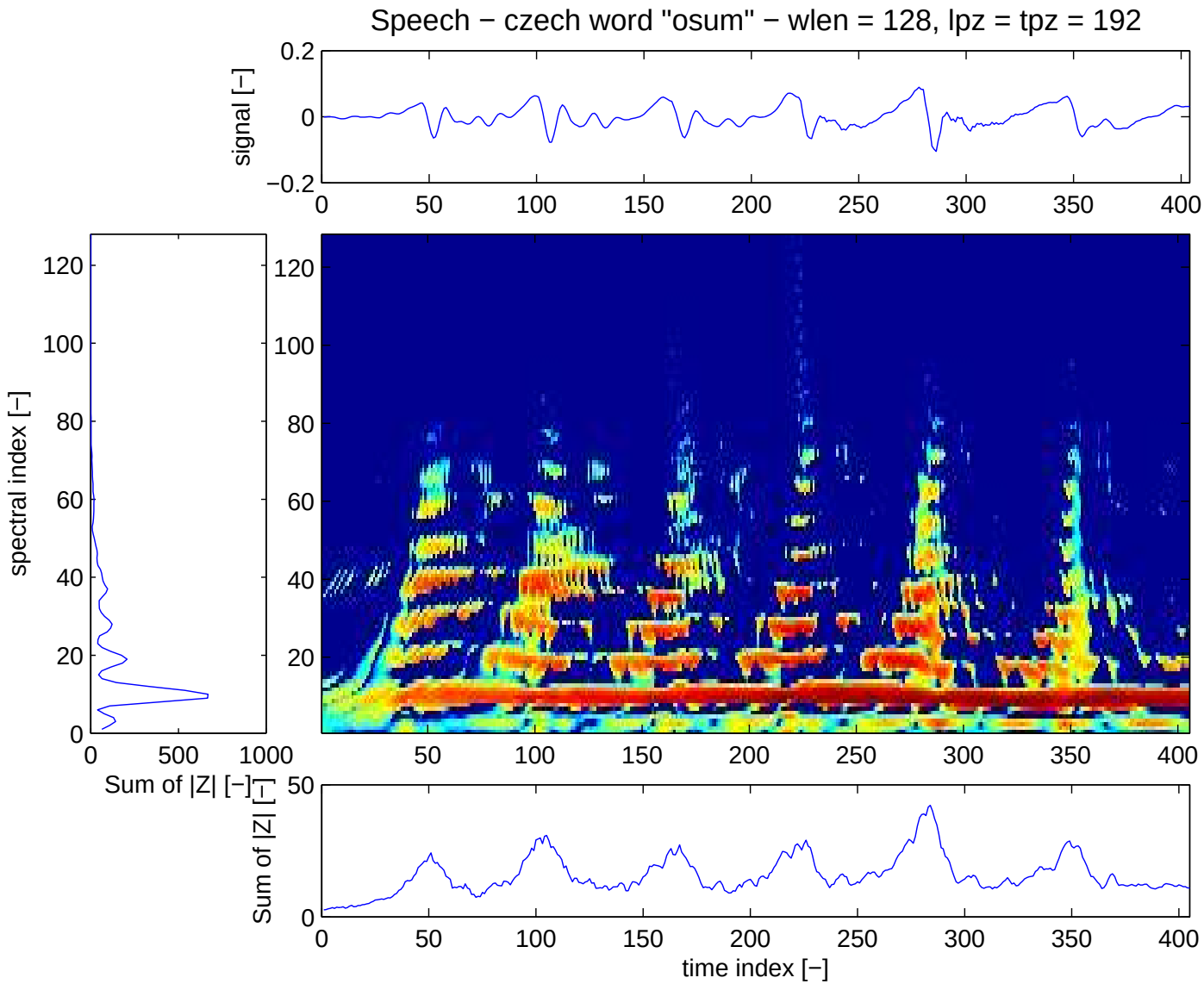
Speech - STADZT k-gram with added leading and trailing zeros



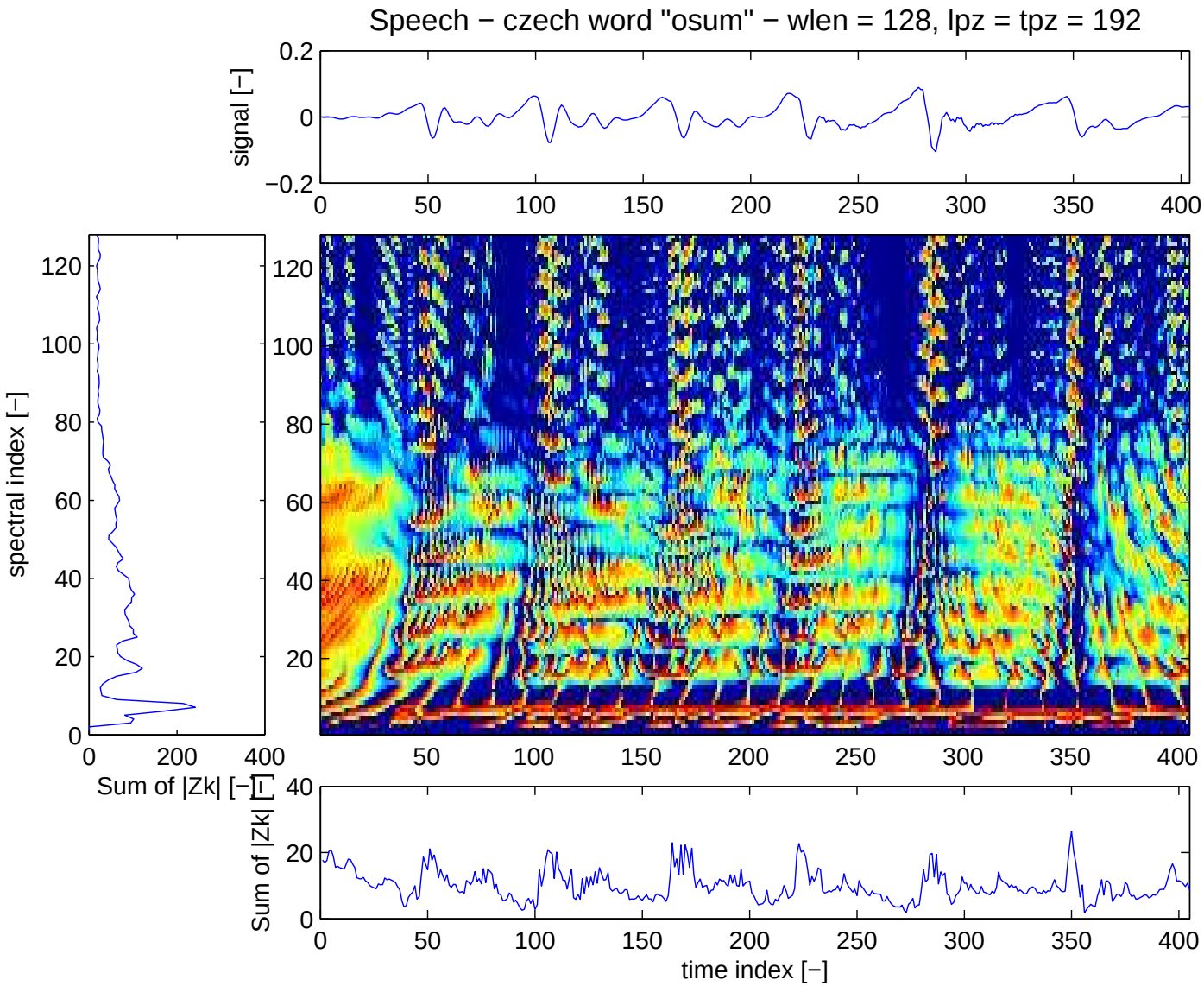
Detail of vowel ”o” - STFT spectrogram



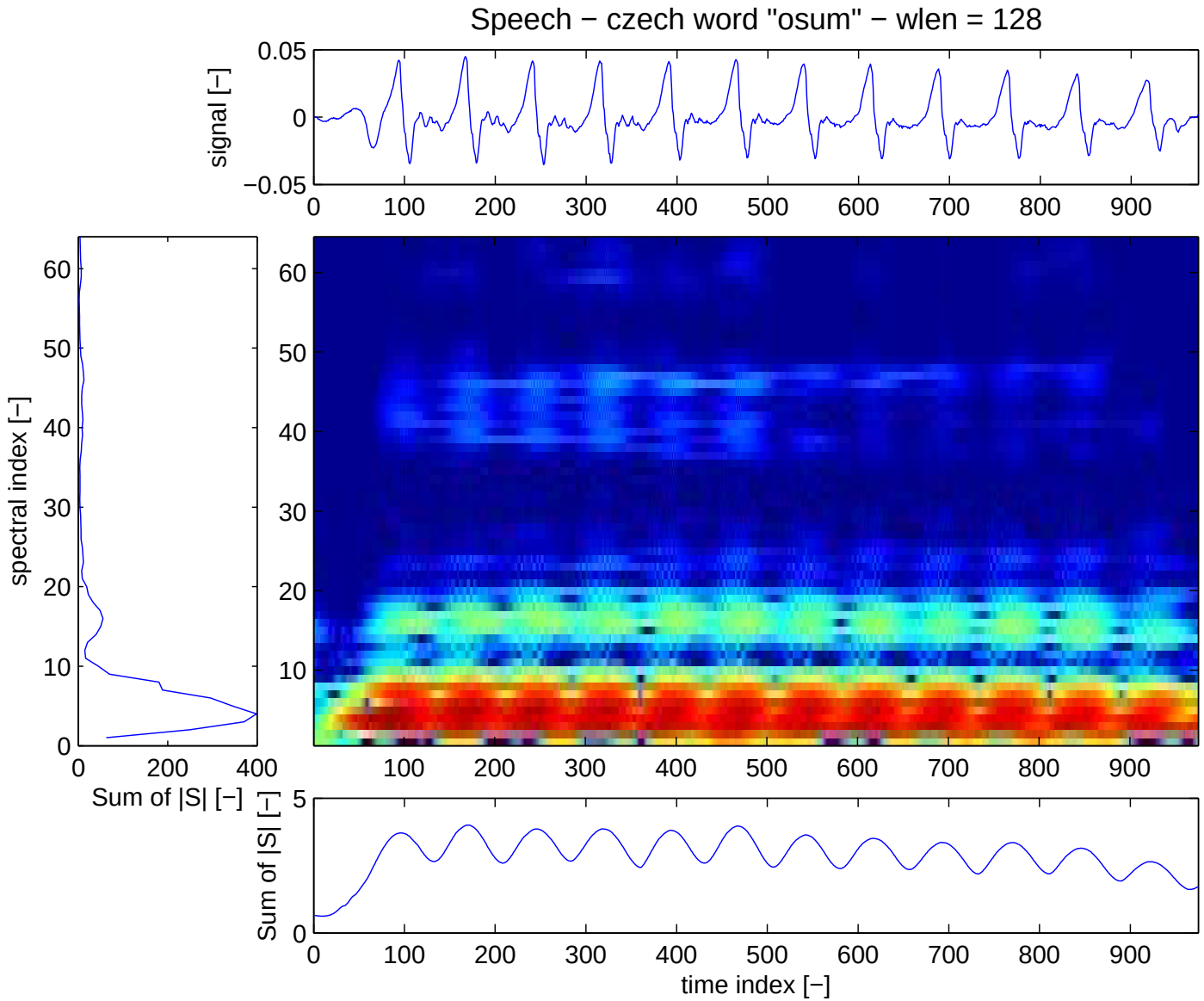
Detail of vowel ”o” - ADZT zologram with added leading and trailing zeros



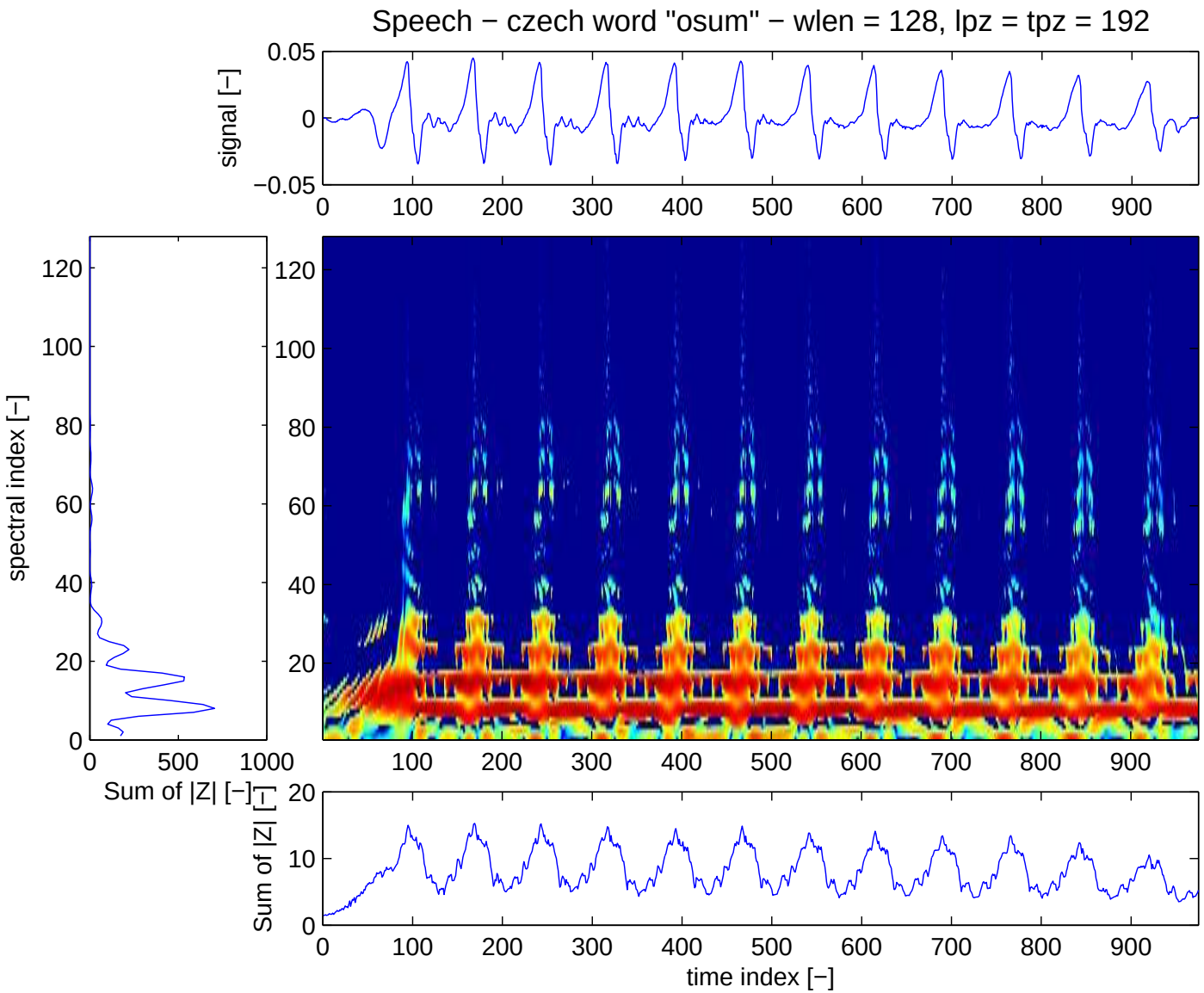
Detail of vowel ”o” - ADZT k-gram with added leading and trailing zeros



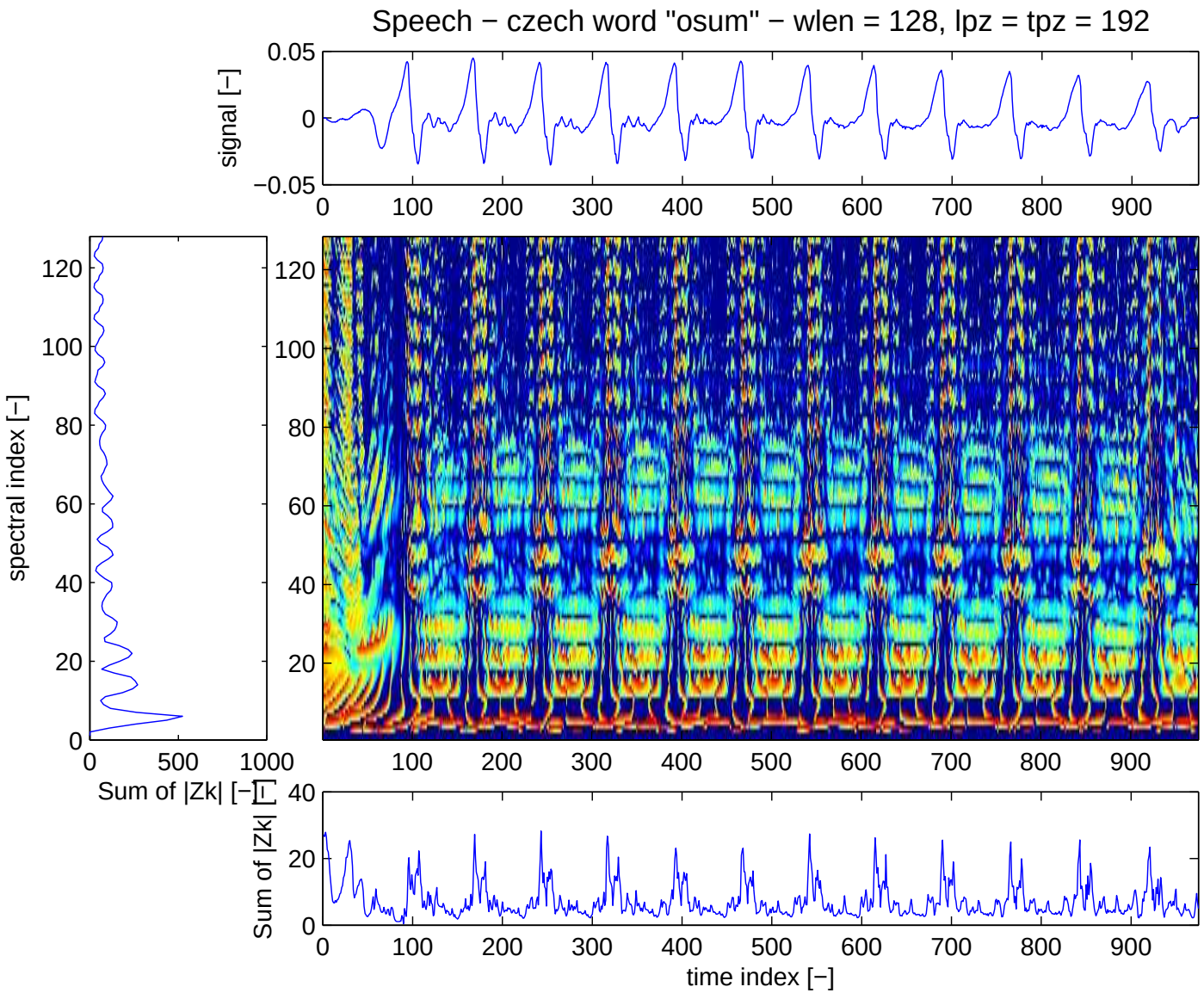
Detail of consonant ”m” - STFT spectrogram



Detail of consonant ”m” - ADZT zologram with added leading and trailing zeros



Detail of consonant ”m” - ADZT k-gram with added leading and trailing zeros



Final remarks

- We perform basic research in the domain *Novel selective transforms for non-stationary signal processing*
- ... recent results are available at http://amber.feld.cvut.cz/selective%20transforms/
- We invite partners who are willing to cooperate with us in particular research projects.