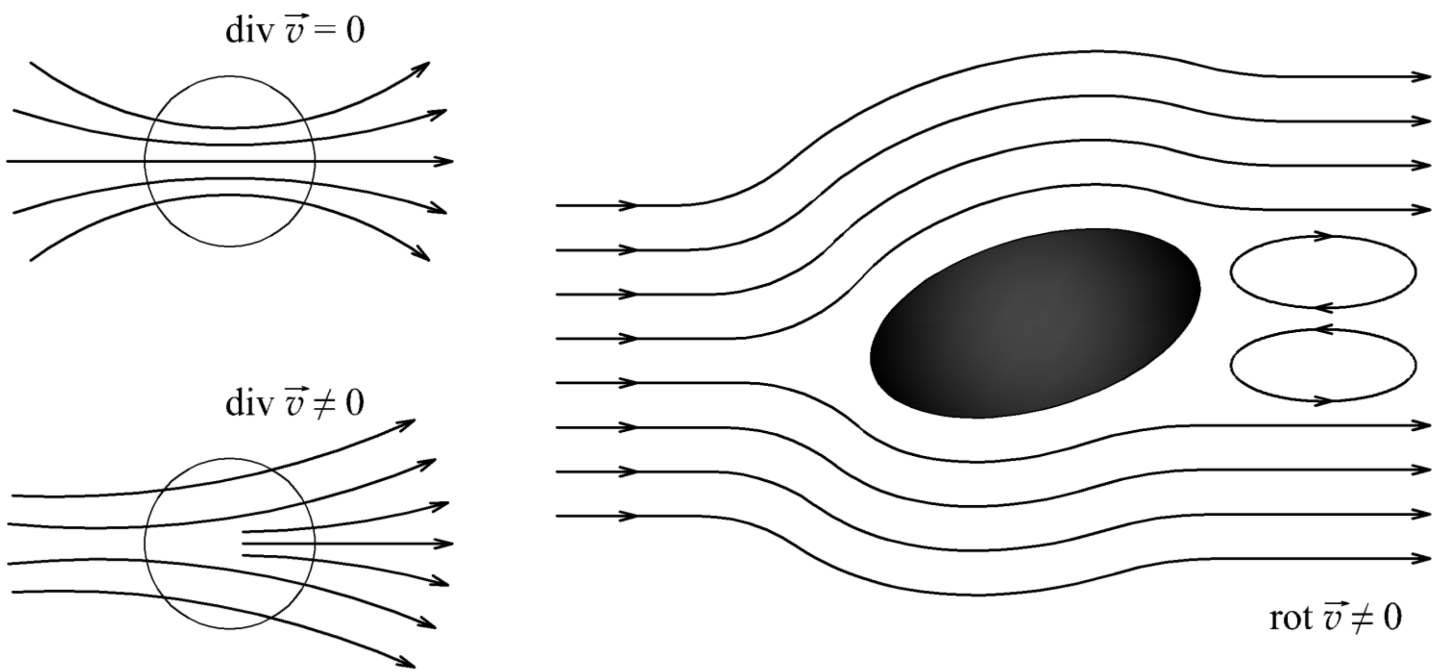
Continuum:
stress and strain



Continuum – study objective

non-rigid bodies – solid, fluid (liquid, gaseous)

problems to solve - strain (deformation)
fluid mechanics
wave propagation

mass point A $\vec{r}_A$ $dm = dm(\vec{r}_A)$ $\vec{v} = \vec{v}(\vec{r}_A)$ $\vec{a} = \vec{a}(\vec{r}_A)$

discrete points → **<u>continuous functions</u>**

$$m = \int dm \qquad \vec{r}_s = \frac{1}{m}\int \vec{r}\, dm \qquad \vec{v}_s = \frac{1}{m}\int \vec{v}\, dm \qquad \vec{a}_s = \frac{1}{m}\int \vec{a}\, dm$$

density of quantities $\rho = \dfrac{dm}{dV}$ $\longrightarrow$ $X_\rho = \dfrac{dX}{dV}$

Continuum descriptions

Define a material point (small volume) and determine its kinematic quantities **Lagrangian description**

X_1, X_2, X_3

movable points

$$x_1 = x_1\left(X_1, X_2, X_3, t\right)$$
$$x_2 = x_2\left(X_1, X_2, X_3, t\right)$$
$$x_3 = x_3\left(X_1, X_2, X_3, t\right)$$

$$T = T(X_1, X_2, X_3, t)$$

Define a point in a fixed coordinate system and describe the kinematic quantities of surrounding continuum, thus, the flow-in and flow-out of the continuum

Eulerian description

x_1, x_2, x_3

fixed points

$$T = T(x_1, x_2, x_3, t)$$

Continuum kinematics

Eulerian description

$$\frac{dT}{dt} = \frac{\partial T}{\partial t}$$ local derivative

Lagrangian description

total derivative

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} v_x + \frac{\partial T}{\partial y} v_y + \frac{\partial T}{\partial z} v_z = \frac{\partial T}{\partial t} + (\vec{v} \cdot \mathrm{grad}\, T)$$

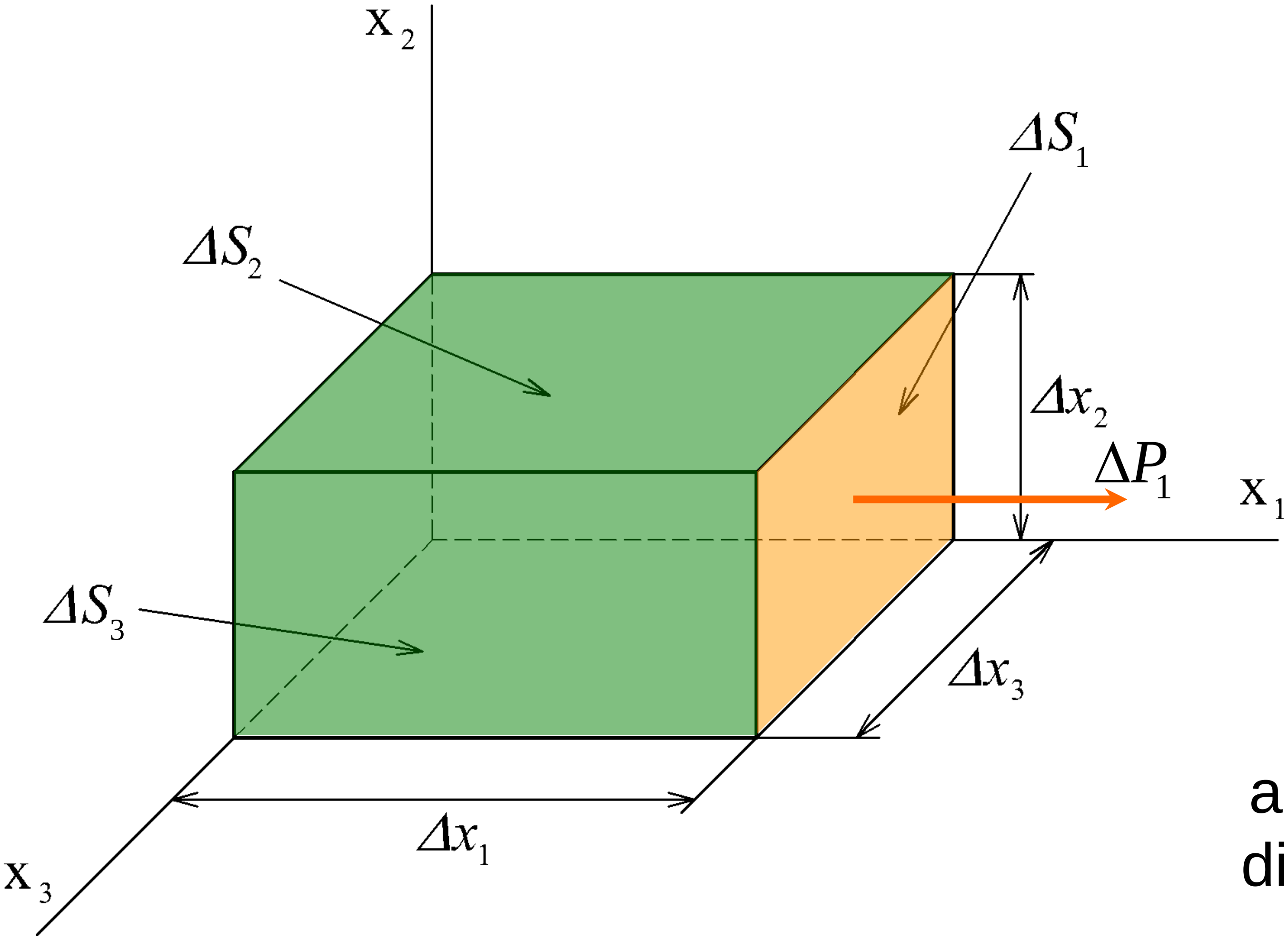
local + convective derivative

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + (\vec{v} \cdot \mathrm{grad}\, T)$$

Continuum dynamics

short-range forces → surface forces

long-range forces → body forces



$$\Delta S_1 = \Delta x_2\ \Delta x_3$$

$$\Delta S_2 = \Delta x_1\ \Delta x_3$$

$$\Delta S_3 = \Delta x_1\ \Delta x_2$$

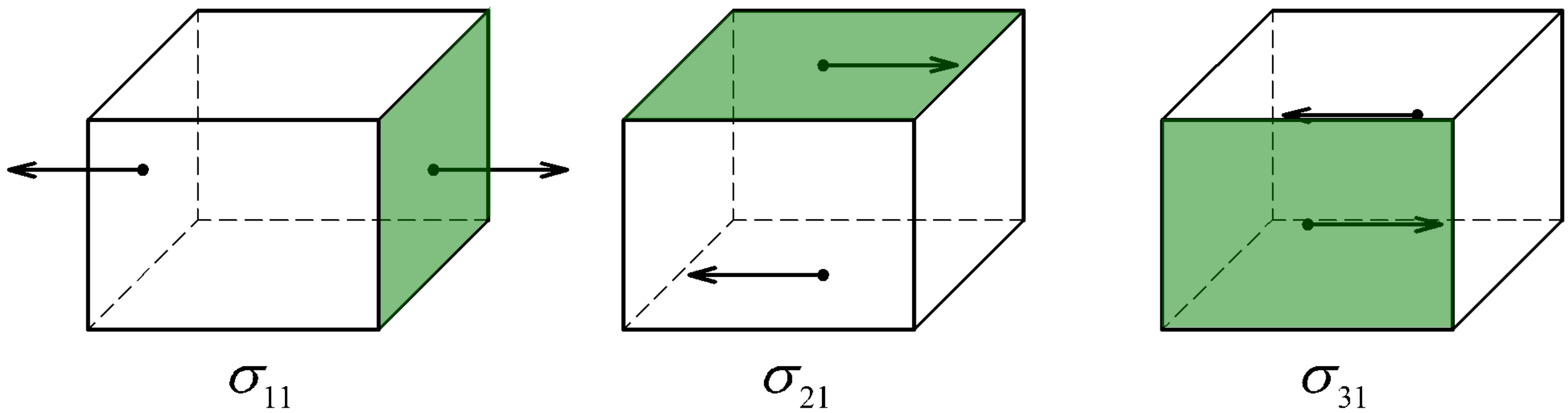
a force in the direction of x_1

Surface forces

pressure (stress) $p = \frac{F}{S}$ $\quad p_1 = \lim_{\Delta S \to 0} \frac{\Delta P_1}{\Delta S_1}$

$$\sigma_{ki} = \lim_{\Delta S_k \to 0} \frac{\Delta P_i}{\Delta S_k}$$

stress tensor

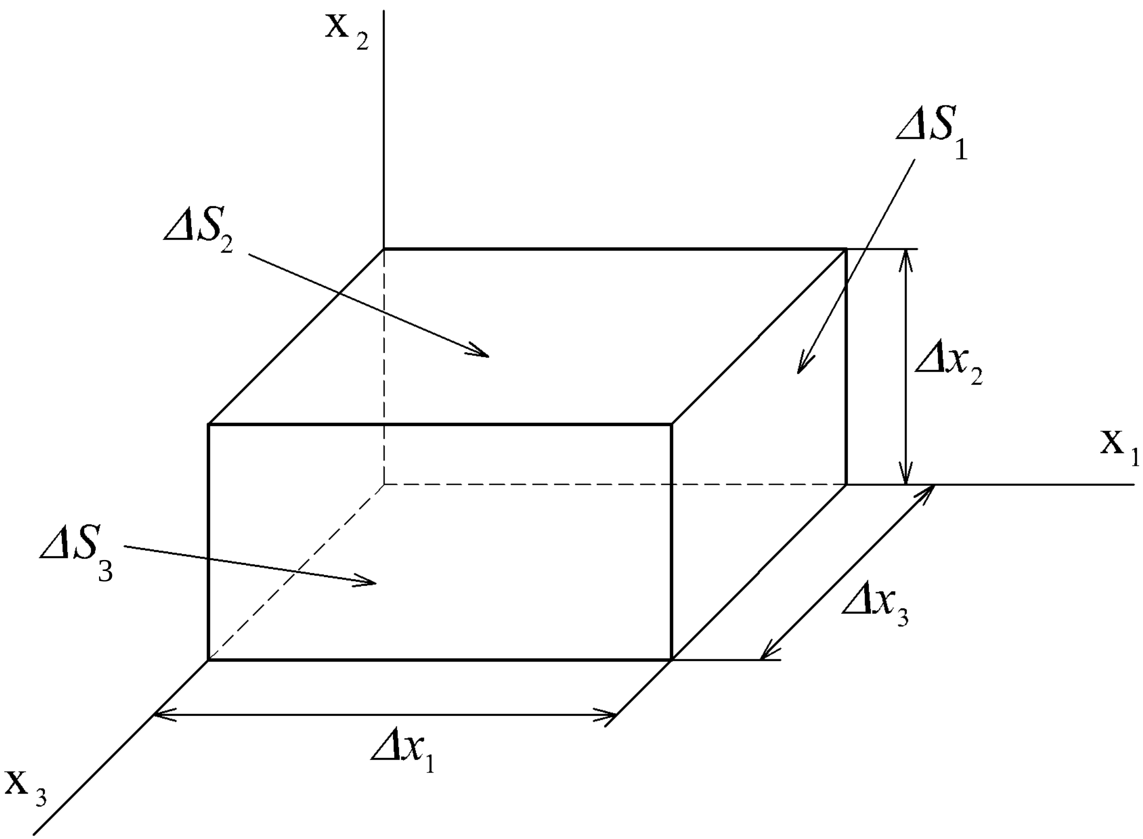


normal tangential tangential

Surface force density

$$\vec{f} = \lim_{\Delta V \to 0} \frac{\Delta \vec{P}}{\Delta V}$$

$$f_1 = \lim_{\Delta V \to 0} \frac{\Delta P_1}{\Delta V} = \lim_{\substack{\Delta x_1 \to 0 \\ \Delta S_1 \to 0}} \frac{\Delta P_1}{\Delta x_1 \Delta S_1}$$



$$f_1 = \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3}$$

$$f_i = \sum_{k=1}^{3} \frac{\partial \sigma_{ki}}{\partial x_k}$$

static pressure

$$-p = \sigma_{11} = \sigma_{22} = \sigma_{33} \qquad f_i = -\frac{\partial p}{\partial x_i} = -\operatorname{grad}\ p$$

body deformation

vector of displacement $\vec{u} = \vec{r}\,' - \vec{r} \qquad \left(u_i = x_i' - x_i\right)$

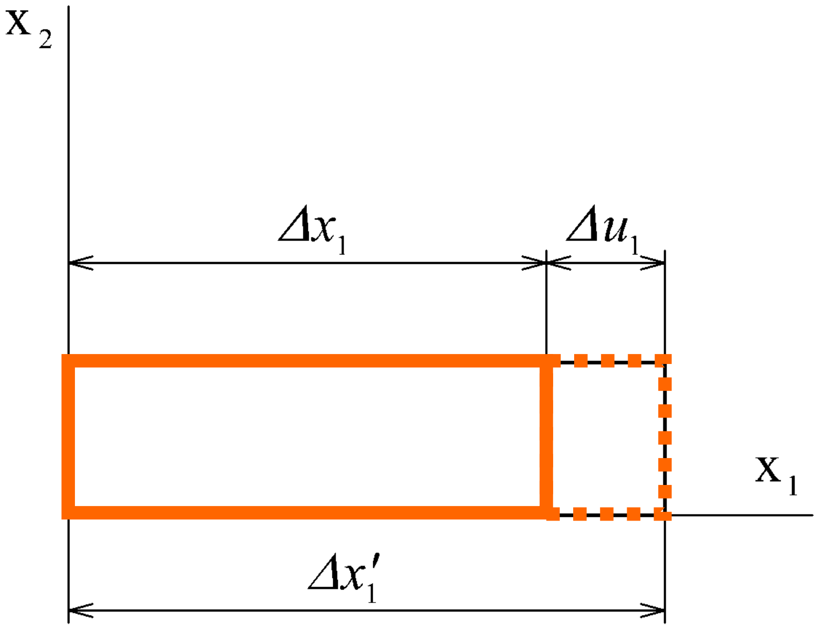
relative displacement

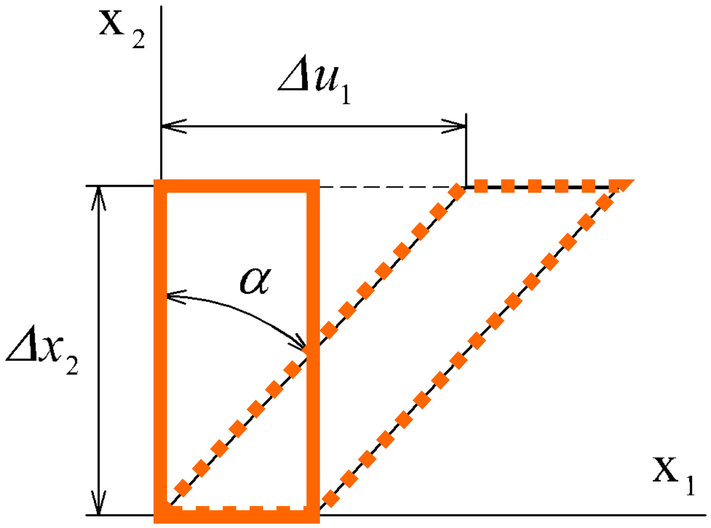
(elongation, extension)

$$\varepsilon = \frac{l' - l}{l}$$

$$\varepsilon_{ii} = \frac{\Delta x_i\,' - \Delta x_i}{\Delta x_i} = \frac{\Delta u_i}{\Delta x_i}$$

Strain (deformation)





Hooke's law $\sigma = E\varepsilon$

$$\varepsilon = \frac{\Delta u_1}{\Delta x_1} \qquad \varepsilon = \frac{\partial u_1}{\partial x_1} \qquad \varepsilon_i = \frac{\partial u_i}{\partial x_i} \qquad \sigma_{ii} = E\frac{\partial u_i}{\partial x_i} \qquad \sigma_{ik} = G\frac{\partial u_k}{\partial x_i}$$

elastic modulus *E*, shear modulus *G*

E* > *G ***G* = 0** for fluids

Body forces

Force of field $\Delta\vec{F}(\vec{r},t)$

Field intensity $\vec{K}(\vec{r},t)=\lim_{\Delta m\to 0}\dfrac{\Delta\vec{F}(\vec{r},t)}{\Delta m}$

Body force density $\vec{f}_0(\vec{r},t)=\lim_{\Delta V\to 0}\dfrac{\Delta\vec{F}(\vec{r},t)}{\Delta V}$

$$\vec{f}_0=\rho\,\vec{K}$$

non-rigid body equilibrium

$$\sum_{k=1}^{3}\frac{\partial\sigma_{ki}}{\partial x_k}+f_{i0}=0\quad,\quad i=1,2,3$$