Seminary exercise Nr. 6
Rigid body

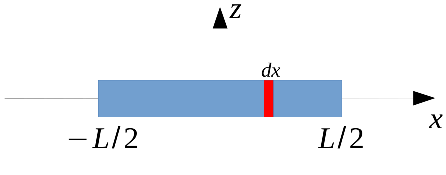
1. A thin, uniform rod has a mass M and a length L. What is the moment of inertia of the rod about the axis being perpendicular to the rod length and passing through its centre of gravity?

thin rod

M

L

$I=?$



$$\rho=\frac{M}{L} \quad ; \quad I_z=\rho\int_{volume} r^2 dV=\rho\int_{-L/2}^{L/2} x^2 dx=\rho\left[\frac{L^3}{24}-\left(-\frac{L^3}{24}\right)\right]=\rho\frac{L^3}{12}=\frac{1}{12}ML^2$$

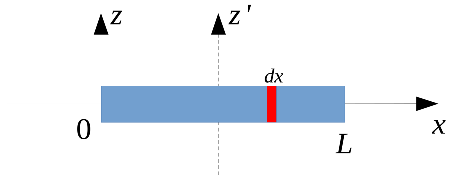
2. A thin, uniform rod has a mass M and a length L. What is the moment of inertia of the rod about the axis being perpendicular to the rod length and passing through the end of the rod? Check the result using the parallel axis theorem and the solution of problem Nr.1.

thin rod

M

L

$I=?$



$$\rho=\frac{M}{L} \quad ; \quad I_z=\rho\int_{volume} r^2 dV=\rho\int_{0}^{L} x^2 dx=\rho\left[\frac{L^3}{3}-\left(-\frac{0^3}{3}\right)\right]=\rho\frac{L^3}{3}=\frac{1}{3}ML^2$$

$$I_z=I_{z'}+M d_{zz'}^2=\frac{1}{12}ML^2+M\left(\frac{L}{2}\right)^2=\frac{1}{3}ML^2$$

7. A gyroscope with momentum of inertia of $0.2\,kg\,m^2$ rotates at a constant speed of $1800\,min^{-1}$. Due to an acting torque, the speed of the gyroscope increases to $3000\,min^{-1}$ within $2.4\,s$. Find the expression of the acting torque considering the fact that the torque is linearly decreasing from its maximal value at $t_0=0\,s$ to zero at $t_1=2.4\,s$.

$I=0.2\,kg\,m^2$

$f_0=1800\,min^{-1}= =30\,Hz$

$f_1=3000\,min^{-1}= =50\,Hz$

$t_0=0\,s$

$t_1=2.4\,s$

τ_0 is max

$\tau_1=0$

$\tau(t)\propto -t$

$\tau(t)=?$

$$\tau(t)=A-Bt \quad ; \quad B>0 \quad ; \quad \tau(t_1)=A-Bt_1=0 \quad ; \quad A=Bt_1 \quad ; \quad A>0$$

$$\tau=\frac{dL}{dt} \quad ; \quad dL=\tau dt \quad ; \quad \int_{L_0}^{L_1} dL=\int_{t_0}^{t_1}\tau dt=\int_{t_0}^{t_1} A-Bt\,dt$$

$$L_1-L_0=A(t_1-t_0)-\frac{B}{2}(t_1^2-t_0^2)=At_1-\frac{1}{2}Bt_1^2=\frac{1}{2}Bt_1^2$$

$$L_0=I\omega_0=I\,2\pi f_0=0.2\,kg\,m^2\cdot 2\pi\cdot 30\,Hz=37.7\,kg\,m^2 s^{-1}$$

$$L_1=I\omega_1=I\,2\pi f_1=0.2\,kg\,m^2\cdot 2\pi\cdot 50\,Hz=62.8\,kg\,m^2 s^{-1}$$

$$B=2\frac{L_1-L_0}{t_1^2}=2\frac{62.8\,kg\,m^2 s^{-1}-37.7\,kg\,m^2 s^{-1}}{(2.4\,s)^2}=8.72\,N\,m\,s^{-1}$$

$$A=Bt_1=8.72\,N\,m\,s^{-1}\cdot 2.4\,s=20.9\,N\,m \quad ; \quad \tau(t)=20.9\,N\,m-8.72\,N\,m\,s^{-1}\cdot t$$

9. A ball ($I=\frac{2}{5}MR^2$) of mass $m=50\,g$ and diameter $d=3\,cm$ rolls smoothly down an incline that leads into a cylindrical loop of radius $r=50\,cm$. From what height h must the ball go down to pass through the entire circular loop of the cylindrical surface?

ball

($I=\frac{2}{5}MR^2$)

$m=50g=0.05\,kg$

$d=3cm=0.03\,m$

$d=50cm=0.5\,m$

$h=?$

$$K_{rolling}=K_{CoM}^{tra}+K_{CoM}^{rot}=\frac{1}{2}mv_{CoM}^2+\frac{1}{2}I_{CoM}\,\omega^2 \quad ; \quad \omega=\frac{v_{CoM}}{d/2} \quad ; \quad I_{CoM}=I_{ball}$$

at the top of the loop: $mg=ma_{CoM}=m\frac{v_{CoM}^2}{r}$

$$v_{CoM}=\sqrt{gr}=\sqrt{9.81\,m\,s^{-2}\cdot 0.5\,m}=2.21\,m\,s^{-1}$$

$$\omega=\frac{2v_{CoM}}{d}=\frac{2\cdot 2.21\,m\,s^{-1}}{0.03\,m}=148\,rad\,s^{-1}$$

$$I_{CoM}=\frac{2}{5}m\left(\frac{d}{2}\right)^2=\frac{2}{5}\cdot 0.05\,kg\cdot\left(\frac{0.03\,m}{2}\right)^2=4.50\cdot 10^{-6}\,kg\,m^2$$

$$U_0=U_{top}+K_{top}^{tra}+K_{top}^{rot} \quad ; \quad mgh=mg2r+\frac{1}{2}mv_{CoM}^2+\frac{1}{2}I_{CoM}\,\omega^2$$

$$h=2r+\frac{v_{CoM}^2}{2g}+\frac{I_{CoM}\,\omega^2}{2mg}=2\cdot 0.5\,m+\frac{(2.21\,m\,s^{-1})^2}{2\cdot 9.81\,m\,s^{-2}}+\frac{4.50\cdot 10^{-6}\,kg\,m^2\cdot(148\,rad\,s^{-1})^2}{2\cdot 0.05\,kg\cdot 9.81\,m\,s^{-2}}=$$
$$=1.35\,m$$

10. What is the kinetic energy of a hoop ($I=MR^2$) rolling smoothly on a horizontal path? The hoop diameter is $50\,cm$, its mass is $2\,kg$ and the linear velocity of the hoop centre is $2\,m\,s^{-1}$.

hoop

($I=MR^2$)

$d=50cm=0.5\,m$

$m=2\,kg$

$v_{CoM}=2\,m\,s^{-1}$

$K=?$

$$K=K_{CoM}^{tra}+K_{CoM}^{rot}=\frac{1}{2}mv_{CoM}^2+\frac{1}{2}I_{CoM}\,\omega^2=\frac{1}{2}mv_{CoM}^2+\frac{1}{2}m(d/2)^2\left(\frac{v_{CoM}}{d/2}\right)^2=$$
$$=mv_{CoM}^2=2\,kg\cdot(2\,m\,s^{-1})^2=8\,J$$