Seminary exercise Nr. 11
Thermodynamics II – First and Second Law, Heat Engines

For all exercises, let assume the following values:

- **gravity acceleration** $g = 9.81\, m\, s^{-2}$
- **gas constant** $R = 8.31\, J\, mol^{-1}\, K^{-1}$

1. An air bubble of volume $20\, cm^3$ is at the bottom of a lake $40\, m$ deep, where the temperature is $4\,°C$. The bubble rises to the surface, which is at a temperature of $20\,°C$. Take the temperature of the bubble's air to be the same as that of the surrounding water. Just as the bubble reaches the surface, what is its volume? The density of water is $\rho = 1000\, kg\, m^{-3}$.

$V_1 = 20\, cm^3 = 2 \cdot 10^{-5}\, m^3$

$T_1 = 4\,°C = 277\, K$

$T_2 = 20\,°C = 293\, K$

$h = 40\, m$

$\rho = 1000\, kg\, m^{-3}$

$V_2 = ?$

$$p_1 = p_{atm} + p_{hydro} = p_{atm} + \rho h g = 1.01 \cdot 10^5\, Pa + 1000\, kg\, m^{-3} \cdot 40\, m \cdot 9.81\, m\, s^{-2} = 4.94 \cdot 10^5\, Pa$$

$$p_2 = p_{atm} = 1.01 \cdot 10^5\, Pa$$

$$p_1 V_1 = n R T_1 \quad ; \quad n = \frac{p_1 V_1}{R T_1} = \frac{4.94 \cdot 10^5\, Pa \cdot 2 \cdot 10^{-5}\, m^3}{8.31\, J\, mol^{-1}\, K^{-1} \cdot 277\, K} = 4.29 \cdot 10^{-3}\, mol$$

$$V_2 = \frac{n R T_2}{p_2} = \frac{4.29 \cdot 10^{-3}\, mol \cdot 8.31\, J\, mol^{-1}\, K^{-1} \cdot 293\, K}{1.01 \cdot 10^5\, Pa} = 1.03 \cdot 10^{-4}\, m^3$$

2. The temperature of $2\, mol$ of an ideal monoatomic gas is raised by $15\,°C$. What are the work done on the gas, the total energy transferred as heat, the change in the internal energy of the gas? Suppose the process is done under i) constant volume and ii) constant pressure.

$n = 2\, mol$

monoatomic gas

$\Delta T = 15\, K$

$W = ?$

$Q = ?$

$\Delta U = ?$

i) V const.

ii) p const.

i) $\Delta V = 0 \quad ; \quad W = -p \Delta V = 0 \quad ; \quad \Delta U = Q \quad ; \quad c_V = \frac{3}{2} R$

$$Q = n c_V \Delta T = 2\, mol \cdot \frac{3}{2} \cdot 8.31\, J\, mol^{-1}\, K^{-1} \cdot 15\, K = 374\, J \quad ; \quad \Delta U = 374\, J$$

ii) $c_p = \frac{5}{2} R \quad ; \quad Q = n c_p \Delta T = 2\, mol \cdot \frac{5}{2} \cdot 8.31\, J\, mol^{-1}\, K^{-1} \cdot 15\, K = 623\, J$

$$W = -p \Delta V = -n R \Delta T = -2\, mol \cdot 8.31\, J\, mol^{-1}\, K^{-1} \cdot 15\, K = -249\, J$$

$$\Delta U = Q + W = 623\, J - 249\, J = 374\, J$$

4. When $20\,J$ are added as heat to $2\cdot 10^{-3}\,mol$ of an ideal gas, the volume changed from $50\,cm^3$ to $100\,cm^3$ while the pressure remained at normal atmospheric pressure. By how much did the internal energy of the gas change? What is the total energy transferred as work? Find the values of c_p and c_V .

isobaric process

$Q=20\,J$

$n=2\cdot 10^{-3}\,mol$

$V_1=50\,cm^3=$
$=5\cdot 10^{-5}\,m^3$

$V_2=100\,cm^3=$
$=1\cdot 10^{-4}\,m^3$

$p=1\,atm=$
$=1.01\cdot 10^5\,Pa$

$\Delta U=?$

$W=?$

$c_p=?$

$c_V=?$

$$Q=n c_p \Delta T \quad ; \quad p\Delta V=nR\Delta T \quad ; \quad \Delta T=\frac{p}{nR}\Delta V \quad ; \quad Q=\frac{c_p p}{R}\Delta V$$

$$c_p=\frac{QR}{p\Delta V}=\frac{20\,J\cdot 8.31\,J\,mol^{-1}K^{-1}}{1.01\cdot 10^5\,Pa\cdot(10^{-4}\,m^3-5\cdot 10^{-5}\,m^3)}=32.9\,J\,mol^{-1}K^{-1}$$

$$c_V=c_p-R=32.9\,J\,mol^{-1}K^{-1}-8.31\,J\,mol^{-1}K^{-1}=24.6\,J\,mol^{-1}K^{-1}$$

$$W=-p\,\Delta V=-1.01\cdot 10^5\,Pa\cdot(10^{-4}\,m^3-5\cdot 10^{-5}\,m^3)=-5.05\,J$$

$$\Delta U=Q+W=20\,J-5.05\,J=15.0\,J$$

6. An ideal diatomic gas undergoes an adiabatic compression. Its initial pressure and volume are $1.2\cdot 10^5\,Pa$ and $0.2\,m^3$ respectively. Its final pressure is $2.4\cdot 10^5\,Pa$. How much work is done on the gas?

diatomic gas

adiabatic process

$p_1=1.2\cdot 10^5\,Pa$

$V_1=0.2\,m^3$

$p_2=2.4\cdot 10^5\,Pa$

$W=?$

$$pV^{\gamma}=const. \quad ; \quad \gamma=\frac{c_p}{c_V}=\frac{\frac{7}{2}R}{\frac{5}{2}R}=\frac{7}{5}$$

$$p_1V_1^{\gamma}=p_2V_2^{\gamma} \quad ; \quad V_2=\left(\frac{p_1}{p_2}\right)^{\frac{1}{\gamma}}V_1=\left(\frac{1.2\cdot 10^5\,Pa}{2.4\cdot 10^5\,Pa}\right)^{\frac{5}{7}}0.2\,m^3=0.122\,m^3$$

$$p_1V_1^{\gamma}=pV^{\gamma} \quad ; \quad p=p_1V_1^{\gamma}V^{-\gamma}$$

$$W=-\int_{V_1}^{V_2} p\,dV=-\int_{V_1}^{V_2} p_1V_1^{\gamma}V^{-\gamma}dV=-p_1V_1^{\gamma}\int_{V_1}^{V_2} V^{-\gamma}dV=-p_1V_1^{\gamma}\left[\frac{1}{1-\gamma}V^{1-\gamma}\right]_{V_1}^{V_2}=$$

$$=-\frac{p_1V_1^{\gamma}}{1-\gamma}\left(V_2^{1-\gamma}-V_1^{1-\gamma}\right)=-\frac{1}{1-\gamma}\left(p_1V_1^{\gamma}V_2^{1-\gamma}-p_1V_1^{\gamma}V_1^{1-\gamma}\right)=$$

$$=-\frac{1}{1-\gamma}\left(p_2V_2^{\gamma}V_2^{1-\gamma}-p_1V_1^{\gamma}V_1^{1-\gamma}\right)=-\frac{p_2V_2-p_1V_1}{1-\gamma}=$$

$$=-\frac{2.4\cdot 10^5\,Pa\cdot 0.122\,m^3-1.2\cdot 10^5\,Pa\cdot 0.2\,m^3}{1-\frac{7}{5}}=13200\,J$$

9. Imagine a Carnot engine that operates between the temperatures $T_H=850\,K$ and $T_L=300\,K$. What is the efficiency of this engine? What is the condition that assures the efficiency to reach 100 %?

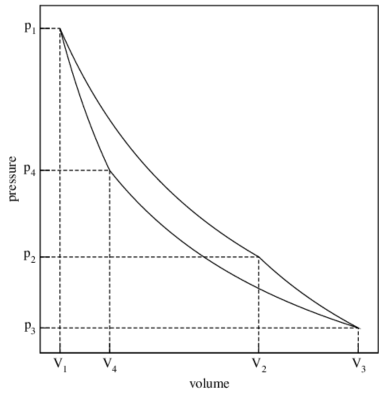
Carnot engine

$T_H=850\,K$

$T_L=300\,K$

$\eta=?$

condition for $\eta=1$?



$1\rightarrow 2$) isothermal expansion

$$p_1V_1=p_2V_2 \quad ; \quad \Delta U_{1\rightarrow 2}=0 \quad ; \quad -Q_{1\rightarrow 2}=W_{1\rightarrow 2}=-nRT_H\ln\frac{V_2}{V_1}$$

$2\rightarrow 3$) adiabatic expansion

$$p_2V_2^{\gamma}=p_3V_3^{\gamma} \quad ; \quad Q_{2\rightarrow 3}=0 \quad ; \quad \Delta U_{2\rightarrow 3}=W_{2\rightarrow 3}=-\frac{nR}{1-\gamma}(T_L-T_H)$$

$3\rightarrow 4$) isothermal compression

$$p_3V_3=p_4V_4 \quad ; \quad \Delta U_{3\rightarrow 4}=0 \quad ; \quad -Q_{3\rightarrow 4}=W_{3\rightarrow 4}=-nRT_L\ln\frac{V_4}{V_3}$$

$4\rightarrow 1$) adiabatic compression

$$p_4V_4^{\gamma}=p_1V_1^{\gamma} \quad ; \quad Q_{4\rightarrow 1}=0 \quad ; \quad \Delta U_{4\rightarrow 1}=W_{4\rightarrow 1}=-\frac{nR}{1-\gamma}(T_H-T_L)$$

$$p_3=p_4\frac{V_4}{V_3} \quad ; \quad p_2V_2^{\gamma}=p_4V_4V_3^{\gamma-1} \quad ; \quad p_2=p_4V_4V_3^{\gamma-1}V_2^{-\gamma}$$

$$p_1=p_2\frac{V_2}{V_1} \quad ; \quad p_4V_4^{\gamma}=p_2V_2V_1^{\gamma-1} \quad ; \quad p_4V_4^{\gamma}=p_4V_4V_3^{\gamma-1}V_2^{-\gamma}V_2V_1^{\gamma-1}$$

$$V_4^{\gamma}=V_4V_3^{\gamma-1}V_2^{-\gamma}V_2V_1^{\gamma-1} \quad ; \quad (V_2V_4)^{\gamma-1}=(V_1V_3)^{\gamma-1} \quad ; \quad V_2V_4=V_1V_3$$

$$\Delta U_{cycle}=\Delta U_{1\rightarrow 2}+\Delta U_{2\rightarrow 3}+\Delta U_{3\rightarrow 4}+\Delta U_{4\rightarrow 1}=-\frac{nR}{1-\gamma}(T_L-T_H)-\frac{nR}{1-\gamma}(T_H-T_L)=0$$

$$W_{cycle}=W_{1\rightarrow 2}+W_{2\rightarrow 3}+W_{3\rightarrow 4}+W_{4\rightarrow 1}=$$

$$=-nRT_H\ln\frac{V_2}{V_1}-\frac{nR}{1-\gamma}(T_L-T_H)-nRT_L\ln\frac{V_4}{V_3}-\frac{nR}{1-\gamma}(T_H-T_L)=$$

$$=-nRT_H\ln\frac{V_2}{V_1}-nRT_L\ln\frac{V_4}{V_3}=-nRT_H\ln\frac{V_2}{V_1}+nRT_L\ln\frac{V_2}{V_1}=-nR(T_H-T_L)\ln\frac{V_2}{V_1}$$

$$\eta=\frac{W_{cycle}}{-Q_{1\rightarrow 2}}=\frac{-nR(T_H-T_L)\ln\frac{V_2}{V_1}}{-nRT_H\ln\frac{V_2}{V_1}}=\frac{T_H-T_L}{T_H}=1-\frac{T_L}{T_H}=1-\frac{300\,K}{850\,K}=0.647$$

$$\eta\rightarrow 100\,\%\Rightarrow T_L\rightarrow 0\vee T_H\rightarrow\infty$$