Experiment Nr. 8

DETERMINATION OF THE ACCELERATION DUE TO GRAVITY USING REVERSIBLE PENDULUM

Theoretical part

The particular coordinate system fixed to the Earth's surface could be considered as an inertial reference frame. There are two basic motions that should be taken into account for our case - **revolution** (motion about the sun along elliptical trajectory with approximate tangential velocity of 29.8 km.s^{-1}) and **rotation** (angular motion about a nearly fixed axis).
When calculating typical mechanical problems (consider a short time range), the motion of revolution could be believed to be uniform on a trajectory with high curvature radius (almost straight-line). Considering this, all inertial accelerations for this motion could be omitted.
The motion of rotation generates a centrifugal acceleration that is perpendicular to the axis of rotation (see $\vec{a}_{od}$ vector in Fig. 1). Considering a spherical shape of the Earth, we can conclude that

$$a_{od} = r\omega^2 = R\omega^2 \cos\varphi \,,$$

where ω is the angular velocity of the Earth's rotation.

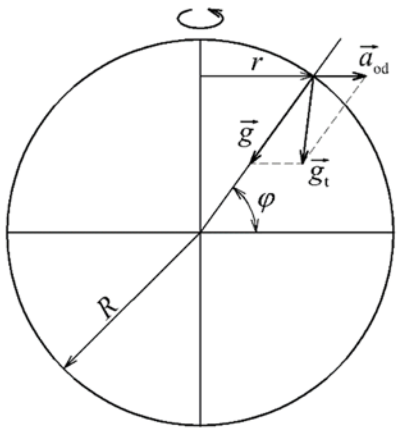

Fig. 1 Centrifugal acceleration on the Earth's surface

The vector of acceleration due to gravity $\vec{g}_t$ consist of sum of two vectors - acceleration caused by gravitation field $\vec{g}$ and centrifugal acceleration $\vec{a}_{od}$ due to the Earth's rotation. The sum of these two vectors is illustrated in Fig. 1 and can be expressed as

$$\vec{g}_t = \vec{g} + \vec{a}_{od}$$

Considering the real shape of the Earth (Earth ellipsoid), the vector of acceleration due to gravity $\vec{g}_t$ is perpendicular to the Earth's surface at every point. The **standard** magnitude of **acceleration due to gravity** is g_t = 9.80665 m.s^{-2}. The approximate value for the location of Prague is g_t = 9.8040 m.s^{-2}.

Reversible pendulum

Reversion pendulum is a special type of the physical pendulum. **Physical pendulum** (Fig. 2) is a rigid body rotating (swinging) around fixed horizontal axis, which do not pass through the

centre of mass of the pendulum T.
A torque acting on the pendulum due to the force of gravity can be evaluated as

$$M = -mga\sin\varphi$$

where m is a mass of the pendulum, a is the distance between the axis of rotation and the centre of mass T and φ is the angular displacement from the equilibrium position. The minus sign means, that the torque acts in the opposite direction compared to the displacement, i.e. it pushes the pendulum back to the equilibrium.
Using the expression of the torque, the motion equation for a body rotating around fixed axis is

$$\frac{d^2\varphi}{dt^2} + \frac{mga}{J}\sin\varphi = 0$$

where J is moment of inertia of the body about the selected axis.

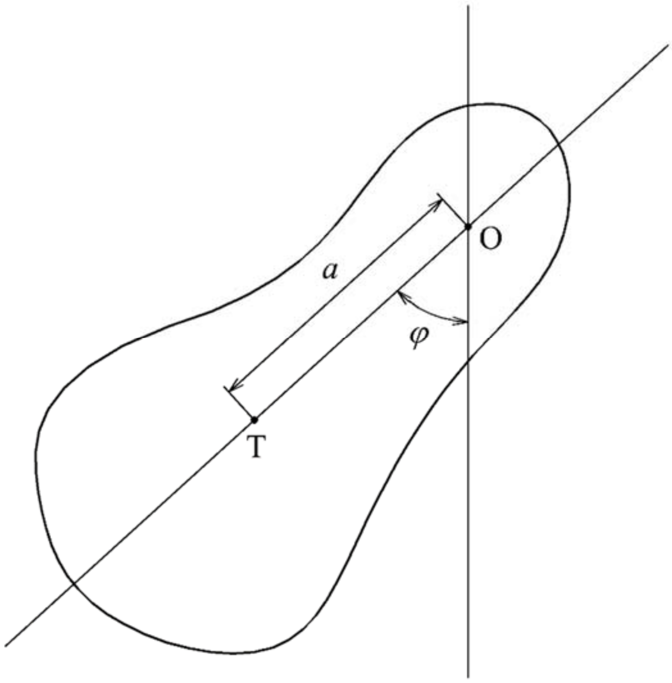

Fig. 2 Physical pendulum scheme

Solution of such nonlinear differential equation leads to an elliptic integral, but we can simplify it by substituting $\sin\varphi \sim \varphi$. Our error would be less than 0.05% for angular displacement smaller than 5°. We obtain simple linear differential equation:

$$\frac{d^2\varphi}{dt^2} + \omega^2\varphi = 0$$

where $\omega^2 = \frac{mga}{J}$ is angular frequency of the pendulum. A swing time (half of the period) is

$$\tau_0 = \frac{T_0}{2} = \pi\sqrt{\frac{J}{mga}}$$

Let's introduce the term **mathematical pendulum** now. It is a particle of mass m hanging on the massless fibre of the length l. The moment of inertia of such pendulum is $J = ml^2$ hence the swing time is now

$$\tau_0 = \pi\sqrt{\frac{ml^2}{mgl}} = \pi\sqrt{\frac{l}{g}}$$

Note that the swing time of mathematical pendulum does not depend on the mass m. Comparing swing time formulae for the physical and mathematical pendulums we can say that the length of the mathematical pendulum is equivalent to the J/md ratio for the physical one.

$$L = \frac{J}{md}$$

The quantity L is **reduced length** of physical pendulum. **Reduced length of the physical pendulum is equal to the length of mathematical pendulum, which has the same swing time**. For the swing time of physical pendulum we can write

$$\tau_0 = \pi\sqrt{\frac{L}{g}}$$

Practically, the **reversible pendulum** is made of a metal bar with two parallel axes O_1 and O_2. A heavy mass m_2 is mounted on the bar and can move along it. The goal is to find special position of the mass m_2 (by measuring its position d from the rod end) so that the swing times are identical for the pendulum hanging on either axis.

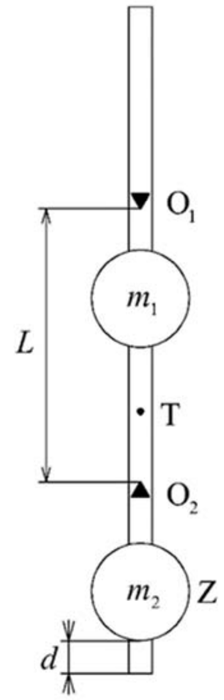

Fig. 3 Reversible pendulum scheme

Measurement objectives

1. Determine the acceleration due to gravity and estimate its uncertainty.
2. Calculate the approximate mass of the Earth and estimate its uncertainty.

Measurement procedure

First, measure the swing times τ_1 and τ_2 for several positions of the mass body m_2. The positions of the mass body m_2 should be selected between 5 and 9 cm from the rod end with a step of 1 cm. The measurement is carried out with the computer assistance. The swing times are registered using an optical equilibrium position sensor recording every pendulum passage. Due to the simplification in the theory, the initial angular displacement has to be lower that 5°.
Fill in the measurement data to the software and calculate the position of the mass body where

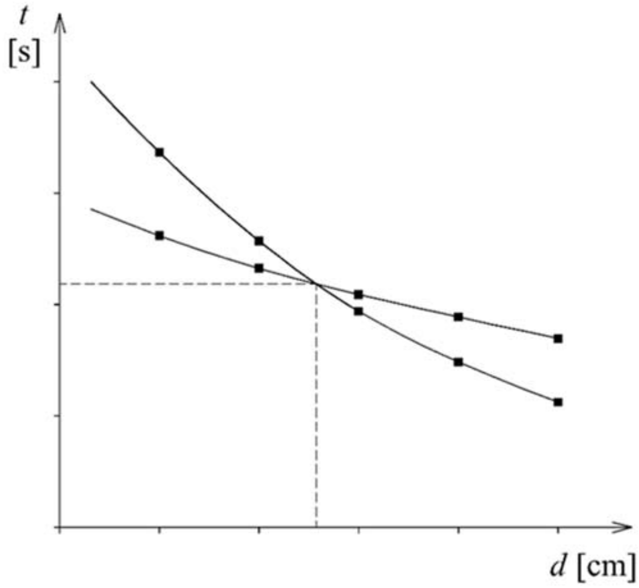

Fig. 4 Typical plot of the swing times dependent on the mass body position

the swing times are the same. The typical graph of two parabolic curves that can be obtained in this measurement, could be seen in Fig. 4.

The position d_0 of the mass body where the swing time is the same correspond to the property that the reduced length L of physical pendulum equals to the length of the mathematical pendulum. In this case, we can conclude that the swing time is

$$\tau_0 = \pi\sqrt{\frac{L}{g}}$$

Make about 10 measurements of the swing times at the d_0 position (5 at the O_1 axis and 5 at the O_2 axis). Use the average time value for further calculation. Determine the data scatter to obtain the time measurement error.
Using this formula, we can calculate the acceleration due to gravity as

$$g = \frac{\pi^2 L}{\tau_0^2}$$

The mass of the Earth can be determined from the Newton's law of gravitation:

$$F_g = mg = \kappa\frac{Mm}{R^2} \qquad \rightarrow \qquad M = \frac{gR^2}{\kappa},$$

where R is the mean Earth radius ($6.38 \cdot 10^6$ m) and κ is the gravitational constant ($6.67 \cdot 10^{-11}$ m^3 kg^{-1} s^{-2}).

Uncertainty calculation notes

Acceleration due to gravity

Using the summation of quadrature law, the relative uncertainty can be expressed as

$$u_{rg} = \sqrt{u_{rLB}^2 + 4u_{r\tau B}^2}$$

where u_{rLB} is the uncertainty of the reduced length of the pendulum (measured by the tape measure) and the $u_{\mathrm{r\tau B}}$ is a scatter of the swing time data obtained from the measurements with the d_0 position of the mass body.

Mass of the Earth

Considering the calculation formula, the uncertainty of the Earth mass can be caused by the uncertainty of the acceleration due to gravity only. The other values can be taken into account as true values (uncertainty equals zero). Thus, the mass relative uncertainty is the same as the relative uncertainty of the acceleration

$$u_{rM} = u_{rg}$$