Experiment Nr. 7

DETERMINATION OF MOMENT OF INERTIA AND SHEAR MODULUS USING TORSIONAL OSCILLATIONS

Theoretical part

The **moment of inertia** J of an object is a calculated quantity for a rigid body that is undergoing rotational motion around a fixed axis. It is calculated based upon the distribution of mass m within the object and the position of the axis (represented by r):

$$J = \int r^2 dm$$

The SI unit of moment of inertia is kg.m^2.
The same object can have very different moment of inertia values depending upon the location and orientation of the axis of rotation. Conceptually, the moment of inertia can be thought of as representing the object's resistance to change in angular velocity, in a similar way to how the mass represents a resistance to the change in velocity in non-rotational motion, under Newton's laws of motion. The torque $\vec{M}$ affecting the rotational motion of an object could be defined by following formula:

$$\vec{M} = \frac{d\vec{L}}{dt} = J\frac{d\vec{\omega}}{dt} = J\vec{\varepsilon},$$

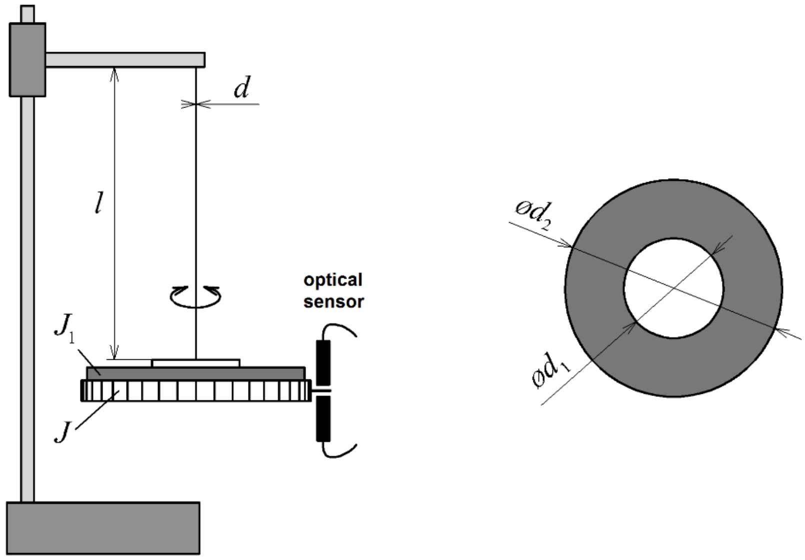
where $\vec{L}$ is the angular momentum, $\vec{\omega}$ is the angular velocity and $\vec{\varepsilon}$ is the angular acceleration.
There is a variety of methods for calculating the moment of inertia that are particularly useful. A number of common objects, such as rotating cylinders or spheres, the calculation by afore-defined integral is relatively easy. There are mathematical means of addressing the problem and calculating the moment of inertia for those objects which are more uncommon and irregular, and thus pose more of a challenge. One of the most suitable method for determination of the moment of inertia for irregular objects is using the concept of oscillations in rotational motion (torsional oscillations).
The **shear modulus** G is one of several quantities for measuring the stiffness of materials. All of them arise in the generalized Hooke's law. The shear modulus describes the material's response to shear stress. Practically, the shear modulus can by determined using torsional oscillations of well-defined torsional pendulum (a mass body fixed on the vertical rod or wire). For the calculation of shear modulus we do need to know the dimensions of the rod / wire and the moment of inertia of the mass body angularly oscillating around a steady position. The measurement objective is the swing time of the oscillations of the torsional pendulum system.

Measurement objectives

1. Determine the moment of inertia J for the irregular object using torsional oscillations and calculate its uncertainty.
2. Determine the shear modulus G of the steel wire and evaluate the measurement uncertainty. Compare the obtained result to the material property given by the tabular data.

Measurement procedure

The torsional pendulum consist of a steel wire (diameter of *d* and the length of *l*) and two mass bodies, see the following figure:



The first mass body has a well-defined moment of inertia J_1 due to its regular shape - it is a ring with inner diameter of d_1, outer diameter of d_2, thickness *t* and mass of m_p. The moment of inertia can be calculated using following formula:

$$J_1 = \frac{1}{8} m_p (d_1^2 + d_2^2)$$

The second mass body has an irregular shape and its moment of inertia should be determined by the by means of its torsional oscillations. Generally, the swing time of the undamped torsional oscillations of a common mass body are given as

$$\tau = \pi \sqrt{\frac{J}{K}}, \text{ where } K = \frac{\pi G d^2}{8l}$$

However, we will use a different approach for determining the *G* value. Let's consider that we know the torsional oscillations swing time τ and τ' for the system of mass bodies with moment of inertia *J* and $J + J_1$. The fraction of the oscillation swing times can be defined as

$$\frac{\tau}{\tau'} = \sqrt{\frac{J}{J+J_1}} \quad \text{and, thus,} \quad J = \frac{J_1}{\left(\frac{\tau'}{\tau}\right)^2 - 1}$$

The shear modulus *G* of the wire can be determined by following formula:

$$G = \frac{32\pi l J}{d^4 \tau^2}$$

The tabular data gives the shear modulus value of steel to be about 80 GPa.

Uncertainty calculation notes

Moment of inertia *J*

The uncertainty consist both of the Type A and Type B. The uncertainty of Type A can be determined as follows

$$u_{JA} = 2J_1 \frac{\tau'}{\tau^2} \frac{1}{\left[\left(\frac{\tau'}{\tau}\right)^2 - 1\right]^2} \sqrt{\left(\frac{\tau'}{\tau}\right)^2 u_{\tau A}^2 + u_{\tau' A}^2} \ ,$$

where the uncertainties of time measurement ($u_{\tau A}$ and $u_{\tau' A}$) correspond to the particular measurement standard deviation or the deviation of the data regression curve.

The uncertainty of Type B estimation do not exceed 1.5 %, thus, it can be considered as

$$u_{rJB} = u_{rJ_1B} = 0.015$$

Shear modulus *G*

The estimation of the Type A uncertainty should consider the sub values of time measurement, moment of inertia and the steel wire diameter, thus

$$u_{rGA} = \sqrt{16u_{rrA}^2 + 4u_{r\tau A}^2 + u_{rJA}^2}$$

The estimation of the Type A uncertainty should consider the sub values of moment of inertia, wire diameter and wire length, thus

$$u_{rGB} = \sqrt{16u_{rrB}^2 + u_{rlB}^2 + u_{rJB}^2}$$

The combined standard deviation of the shear modulus should be calculated by

$$u_G = \sqrt{u_{GA}^2 + u_{GB}^2}, \text{ event. } u_{rG} = \sqrt{u_{rGA}^2 + u_{rGB}^2}$$