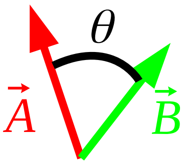
Some solution trace of Seminary exercise Nr. 1
Kinematics of a mass point

1. Define the dot (scalar) and the cross (vector) products of two arbitrary 3D vectors $\vec{A}$ and $\vec{B}$. Find an example in Physics where these products are used.



Dot (scalar) product: $\vec{A}\cdot\vec{B}=\begin{vmatrix}A_1\\A_2\\A_3\end{vmatrix}\cdot\begin{vmatrix}B_1\\B_2\\B_3\end{vmatrix}=A_1B_1+A_2B_2+A_3B_3$

Cross (vector) product:

$$\vec{A}\times\vec{B}=\begin{vmatrix}A_1\\A_2\\A_3\end{vmatrix}\times\begin{vmatrix}B_1\\B_2\\B_3\end{vmatrix}=det\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\A_1 & A_2 & A_3\\B_1 & B_2 & B_3\end{vmatrix}=$$

$$=\vec{i}\,(A_2B_3-A_3B_2)-\vec{j}\,(A_1B_3-A_3B_1)+\vec{k}\,(A_1B_2-A_2B_1)=$$

$$=\begin{vmatrix}A_2B_3-A_3B_2\\A_3B_1-A_1B_3\\A_1B_2-A_2B_1\end{vmatrix}$$

Magnitude of a vector: $\|\vec{A}\|=\sqrt{\vec{A}\cdot\vec{A}}$,

$\vec{A}\cdot\vec{B}=\|\vec{A}\|\cdot\|\vec{B}\|\cos\theta$, $\|\vec{A}\times\vec{B}\|=\|\vec{A}\|\cdot\|\vec{B}\|\sin\theta$

3. Define the unit (base) vector of an arbitrary vector $\vec{A}$.

$$\vec{v}_A=\frac{\vec{A}}{\|\vec{A}\|}=\begin{vmatrix}\dfrac{A_1}{\sqrt{A_1^2+A_2^2+A_3^2}}\\\dfrac{A_2}{\sqrt{A_1^2+A_2^2+A_3^2}}\\\dfrac{A_3}{\sqrt{A_1^2+A_2^2+A_3^2}}\end{vmatrix}$$

5. The vector $\vec{A}$ lies on the *xy*-plane and it forms an angle of 45 degrees with the *x*-axis. Find the components of the vector $\vec{A}$, if $\|\vec{A}\|=2$.

$\vec{A}\in xy$

$\theta=45°$

$\|\vec{A}\|=2$

$A_1=?$

$A_2=?$

$A_3=?$

$$\vec{A}\in xy\Rightarrow\vec{A}\perp\vec{k}\Rightarrow\vec{A}\cdot\vec{k}=0$$

$$\vec{A}\cdot\vec{k}=\begin{vmatrix}A_1\\A_2\\A_3\end{vmatrix}\cdot\begin{vmatrix}0\\0\\1\end{vmatrix}=A_3=0$$

$$\vec{A}\cdot\vec{i}=\begin{vmatrix}A_1\\A_2\\0\end{vmatrix}\cdot\begin{vmatrix}1\\0\\0\end{vmatrix}=A_1=\|\vec{A}\|\cos\theta=2\frac{\sqrt{2}}{2}=\sqrt{2}$$

$$\|\vec{A}\|=2=\sqrt{2+A_2^2}$$

$$2+A_2^2=4$$

$$A_2=\sqrt{2}$$

7. Let the position of a particle on the *x*-axis be $x(t)=3t-4t^2+t^3$. Find the appropriate functions for the velocity and the acceleration of the particle and describe the type of motion. Calculate all kinematic quantities at $t_1=1$, $t_2=2$ and $t_3=5$.

$x(t)=3t-4t^2+t^3$

$v(t)=?$

$a(t)=?$

$t_1=1$

$t_2=2$

$t_3=5$

$$v(t)=\frac{dx(t)}{dt}=\frac{d}{dt}(3t-4t^2+t^3)=3-8t+3t^2$$

$$a(t)=\frac{dv(t)}{dt}=\frac{d}{dt}(3-8t+3t^2)=-8+6t$$

$$\begin{matrix} x(t_1)=0 & & x(t_2)=-2 & & x(t_3)=40 \\ v(t_1)=-2 & ; & v(t_2)=-1 & ; & v(t_3)=38 \\ a(t_1)=-2 & & a(t_2)=4 & & a(t_3)=22 \end{matrix}$$

9. The velocity of a particle is linear with respect to time. Find the appropriate functions of acceleration and position.

$v(t)\propto t$

$a(t)=?$

$x(t)=?$

$$v(t)=A+Bt$$

$$a(t)=\frac{dv}{dt}=\frac{d}{dt}(A+Bt)=B$$

$$\frac{dx}{dt}=v(t) \quad ; \quad dx=v(t)dt \quad ; \quad \int_{x_0}^{x(t)} dx=\int_0^t v(t')dt' \quad ; \quad \int_{x_0}^{x(t)} dx=\int_0^t (A+Bt')dt'$$

$$x(t)-x_0=A(t-0)+\frac{1}{2}B(t^2-0^2) \quad ; \quad x(t)=x_0+At+\frac{1}{2}Bt^2$$

11. The position vector of a particle is given by $\vec{r}(t)=A\cos(3bt)\vec{i}+A\sin(3bt)\vec{j}$, where A and b are constants (explain their physical meaning). Find the components of the velocity and the acceleration vectors and calculate their magnitudes. Describe the type of motion and find the angular speed and period.

$\vec{r}(t)=A\cos(3bt)\vec{i}+$
$+A\sin(3bt)\vec{j}$

$v_x(t)=?$

$v_y(t)=?$

$a_x(t)=?$

$a_y(t)=?$

$\|\vec{v}(t)\|=?$

$\|\vec{a}(t)\|=?$

$\omega=?$

$T=?$

$$x(t)=A\cos(3bt)$$
$$y(t)=A\sin(3bt)$$

$$v_x(t)=\frac{d\,x(t)}{dt}=-3Ab\sin(3bt)$$

$$v_y(t)=\frac{d\,y(t)}{dt}=3Ab\cos(3bt)$$

$$a_x(t)=\frac{d\,v_x(t)}{dt}=-9\,Ab^2\cos(3bt)$$

$$a_y(t)=\frac{d\,v_y(t)}{dt}=-9\,Ab^2\sin(3bt)$$

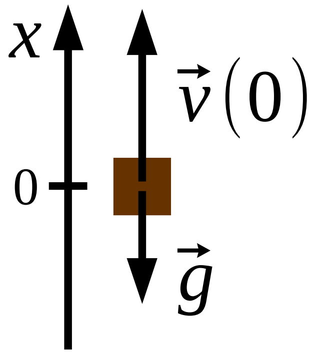
$$\|\vec{v}\|=\sqrt{[-3\,Ab\sin(3bt)]^2+[3\,Ab\cos(3bt)]^2}=3\,Ab$$

$$\|\vec{a}\|=\sqrt{[-9\,Ab^2\cos(3bt)]^2+[-9\,Ab^2\sin(3bt)]^2}=9\,Ab^2$$

$$\begin{matrix} x=r\cos\theta \\ y=r\sin\theta \end{matrix} \quad ; \quad r=\sqrt{x^2+y^2}=A \quad ; \quad r\cos\theta=A\cos(3bt)\Rightarrow\theta=3bt \quad ;$$

$$\omega(t)=\frac{d\,\theta(t)}{dt}=3b \quad ; \quad T=\frac{2\pi r}{\|\vec{v}\|}=\frac{2\pi}{\omega}=\frac{2\pi}{3b}$$

12. A small ball is tossed vertically at a constant initial speed of 12 m s^-1. Find the functions of velocity and position and calculate the maximum theoretic height that can be reached.



$v(0)=12\,m\,s^{-1}$

$x(0)=0\,m$

$g=9.81\,m\,s^{-2}$

$v(t)=?$

$x(t)=?$

$x_{max}=?$

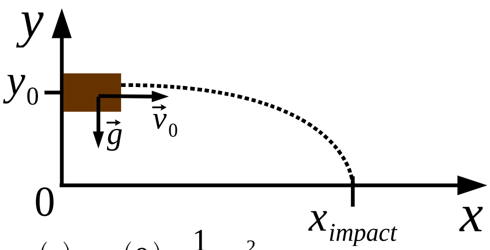
$$v(t)=v(0)-g\,t$$

$$x(t)=v(0)\,t-\frac{1}{2}g\,t^2$$

$$v(t_{max})=0=v(0)-g\,t_{max} \quad ; \quad t_{max}=\frac{v(0)}{g}=\frac{12\,m\,s^{-1}}{9.81\,m\,s^{-2}}=1.22\,s$$

$$x_{max}=v(0)\,t_{max}-\frac{1}{2}g\,t_{max}^2=12\,m\,s^{-1}\cdot 1.22\,s-\frac{1}{2}\cdot 9.81\,m\,s^{-2}\cdot(1.22\,s)^2=7.34\,m$$

13. A rescue plane flies at constant speed of 200 km h^-1 at a height of 0.5 km on the sea level. A rescue bag is dropped in order to fall down directly to the point of a victim location. At which horizontal distance should the bag be dropped? What is the final impact speed of the bag?



$v_x(0)=200\,km\,h^{-1}$
$=55.6\,m\,s^{-1}$

$y(0)=0.5\,km$
$=500\,m$

$y(t_{impact})=0\,m$

$x(0)=0\,m$

$g=9.81\,m\,s^{-2}$

$x(t_{impact})=?$

$\|\vec{v}(t_{impact})\|=?$

$$y(t)=y(0)-\frac{1}{2}g\,t^2$$

$$y(0)-\frac{1}{2}g\,t_{impact}^2=0 \quad ; \quad t_{impact}=\sqrt{\frac{2\,y(0)}{g}}=\sqrt{\frac{2\cdot 500\,m}{9.81\,m\,s^{-2}}}=10.1\,s$$

$$x(t)=v_x(0)\,t \quad ; \quad x(t_{impact})=v_x(0)\,t_{impact}=55.6\,m\,s^{-1}\cdot 10.1\,s=562\,m$$

$$v_y(t)=-g\,t \quad ; \quad v_y(t_{impact})=-g\,t_{impact}=-9.81\,m\,s^{-2}\cdot 10.1\,s=-99.1\,m\,s^{-1}$$

$$v_x(t)=v_x(0) \quad ; \quad v_x(t_{impact})=55.6\,m\,s^{-1}$$

$$\|\vec{v}(t_{impact})\|=\sqrt{v_x(t_{impact})^2+v_y(t_{impact})^2}=\sqrt{(55.6\,m\,s^{-1})^2+(-99.1\,m\,s^{-1})^2}=114\,m\,s^{-1}$$

15. A car is running on a circular track at a constant speed of 150 km h^-1. The curvature radius of the track is 500 m. Calculate the magnitude of the centripetal acceleration.

$\|\vec{v}(0)\|=150\,km\,h^{-1}$
$=41.7\,m\,s^{-1}$

$r=500\,m$

$a=?$

$$\|\vec{a}\|=\frac{\|\vec{v}\|^2}{r}=\frac{(41.7\,m\,s^{-1})^2}{500\,m}=3.48\,m\,s^{-2}$$

16. The circular motion of a particle shows a tangential velocity with constant magnitude (select the invariables of the problem). Define the functions of angular velocity and position and plot all functions in a graph.

$\|\vec{v}\| = v_0$

$\theta(0) = \theta_0$

r

$\omega = ?$

$\theta(t) = ?$

$$\omega = \frac{\|\vec{v}\|}{r} = \frac{v_0}{r} \quad ; \quad \omega = \frac{d\theta}{dt} \quad ; \quad d\theta = \omega dt \quad ; \quad \int_{\theta(0)}^{\theta(t)} d\theta = \int_0^t \omega dt$$

$$\theta(t) - \theta(0) = \omega(t - 0) \quad ; \quad \theta(t) = \theta_0 + \omega t = \theta_0 + \frac{v_0}{r} t$$