Lineární regrese se známým rozptylem šumu

↳ nemáme rádi algebru, KD nám ukáže, jak to jde snadno převodem do Exp. Třídy + konj. apriorem

$$y_t = \beta^T x_t + e_t$$

$$* \; f(y_t|\beta, x_t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{-1}{2\sigma^2}(y_t - \beta^T x)^T (y_t - \beta^T x)\right\}$$

$$\mathcal{N}(\beta^T x_t, \sigma^2) \propto \exp\left\{-\frac{1}{2\sigma^2}\left[y_t^T y_t - x^T\beta^T y - \beta^T x y_t + x^T\beta\beta^T x\right]\right\}$$

$$x_t^T \beta^T y_t = \beta^T x_t y_t$$

$$\propto \exp\left\{-\frac{1}{2\sigma^2}\left[-2\beta^T x_t y_t + \mathrm{Tr}(\beta\beta^T x x^T)\right] + \text{const}\right\}$$

$$\mathrm{Tr}(A^T B) = \mathrm{vec}(A)^T \mathrm{vec}(B)$$

$$\propto \exp\left\{-\frac{1}{2\sigma^2}\left[-2\beta^T x_t y_t + \mathrm{vec}(\beta\beta^T)^T \cdot \mathrm{vec}(xx^T) + \text{const.}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2\sigma^2}\begin{bmatrix}-2\beta \\ \mathrm{vec}(\beta\beta^T)\end{bmatrix}^T \begin{bmatrix}x_t y_t \\ \mathrm{vec}(xx^T)\end{bmatrix} + \text{const.}\right\} \Rightarrow \boxed{\eta = \begin{bmatrix}+\beta \\ -\frac{\mathrm{vec}(\beta\beta^T)}{2}\end{bmatrix}} \quad T = \begin{bmatrix}x_t y_t \sigma^{-2} \\ \mathrm{vec}(xx^T)\sigma^{-2}\end{bmatrix}$$

$$\xi = \begin{bmatrix}\Sigma^{-1} b \\ \mathrm{vec}(\Sigma^{-1})\end{bmatrix}$$

$$p(\beta|x, \xi) \propto \exp\left\{-\frac{1}{2}(\beta - b)^T \Sigma^{-1}(\beta - b)\right\} \quad \mathcal{N}(b, \Sigma)$$

$$\propto \exp\left\{-\frac{1}{2}\left[\beta^T\Sigma^{-1}\beta - b^T\Sigma^{-1}\beta - \beta^T\Sigma^{-1}b + b^T\Sigma^{-1}b\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\mathrm{Tr}(\beta\beta^T\Sigma^{-1}) - 2\mathrm{Tr}(\beta^T\Sigma^{-1}b) + \text{const.}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\begin{bmatrix}-2\beta \\ \mathrm{vec}(\beta\beta^T)\end{bmatrix}^T\begin{bmatrix}\Sigma^{-1}b \\ \mathrm{vec}(\Sigma^{-1})\end{bmatrix} + \text{const.}\right\}$$

$$\propto \exp\left\{\begin{bmatrix}\beta \\ -\frac{\mathrm{vec}(\beta\beta^T)}{2}\end{bmatrix}^T\begin{bmatrix}\Sigma^{-1}b \\ \mathrm{vec}(\Sigma^{-1})\end{bmatrix} + \text{const}\right\}$$

$$\xi' = \xi + T(x, y)$$

$$(\Sigma^{-1})' = \Sigma^{-1} + x_t x_t^T \sigma^{-2}$$

$$(\Sigma^{-1}b)' = \Sigma^{-1}b + x_t y_t \sigma^{-2}$$

$$(b)' = (\Sigma^{-1} + x_t x_t^T \sigma^{-2})^{-1}(\Sigma^{-1}b + x_t y_t \sigma^{-2})$$

Volba $\beta \sim \mathcal{N}(b, \sigma^2\Sigma)$ odstraní σ^2 v T

KALMAN – chybí jen si přepracovat regrese ☺

$$x_t | x_{t-1}, u_t \sim \mathcal{N}(A_t x_{t-1} + B_t u_t, Q_t)$$
$$y_t | x_t \sim \mathcal{N}(H_t x_t, R_t)$$

+ Apriorno $x_{t-1} | u_{0:t-1}, y_{0:t-1} \sim \mathcal{N}(x_{t-1}^+, P_t^+)$

Predikce $p(x_t | u_{0:t-1}, y_{0:t-1}, u_t) = \int p(x_t | x_{t-1}, u_t) \cdot p(x_{t-1} | u_{0:t-1}, y_{0:t-1})\, dx_{t-1}$

→ 2 pohledy < marginalizace „2rozměrné" $\mathcal{N}$ distribuce; transformace náh. veličiny $\sim \mathcal{N}(x_{t-1}^+, P_t^-)$

$$x_t^- = A_t x_{t-1}^+ + B_t u_t$$
$$P_t^- = A_t P_{t-1}^+ A_t^T + Q_t$$

Update: N.B. * – tady ale máme více rozměrné měření $y_t \Rightarrow s / \sigma^2 / R$

↳ $f(y_t | x_t) \propto \exp\{-\frac{1}{2}(y_t - H_t x_t)^T R_t^{-1} (y_t - H_t x_t)\}$

– zdlouhavost: $s / x_t / H_t$, tedy $x_t \equiv \beta_t$! (bo s dim nepřišli stacionárně)

– R_t^{-1} se tedy dostal mezi členy

– $p(x_t | \nu_{t-1}, \xi_{t-1}) \propto \exp\{-\frac{1}{2}(x_t - x_t^-)^T (P_t^-)^{-1} (x_t - x_t^-)\}$

$$\xi_t = \begin{bmatrix} (P_t^-)^{-1} (x_t^-)^+ \\ \mathrm{vec}[(P_t^-)^{-1}] \end{bmatrix}$$

$$T = \begin{bmatrix} H_t^T R_t^{-1} y_t \\ \mathrm{vec}(H_t^T R_t^{-1} H_t) \end{bmatrix}$$

posterior $(\xi)' = \xi + T \Rightarrow$

$$((P_t^-)^{-1})' = (P_t^+)^{-1} = (P_t^-)^{-1} + H_t^T R_t H_t$$

$$x_t^+ = P_t^+ [(P_t^-)^{-1} x_t^- + H_t^T R_t^{-1} y_t]$$

Sekvenční logistická regrese

$E[y_t|x_t,\beta] = g^{-1}(\beta^T x_t)$ ← linková fce GLM

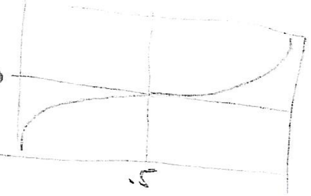
- Rozšíření
- např: neuronové sítě

$y_t \in \{0,1\}, x_t \in \mathbb{R}^n, \beta \in \mathbb{R}^n$ → pst, $p_t \in [0,1]$

$y_t \sim Bernoulli(p_t)$

$f(y_t|p_t) = f(y_t|x_t,\beta) = p_t^{y_t}(1-p_t)^{1-y_t}$ kde p_t je svázáno s prediktorem $\beta^T x_t$ takto:

$$g(p_t) = logit(p_t) = \log\frac{p_t}{1-p_t} = \beta^T x_t$$



$$E[y_t|x_t,\beta] = p_t = logit^{-1}(\beta^T x_t) = \frac{1}{1+e^{-\beta^T x_t}}$$

Nemáme vhodné konjugované apriorno

→ Aproximace (Laplaceovská)

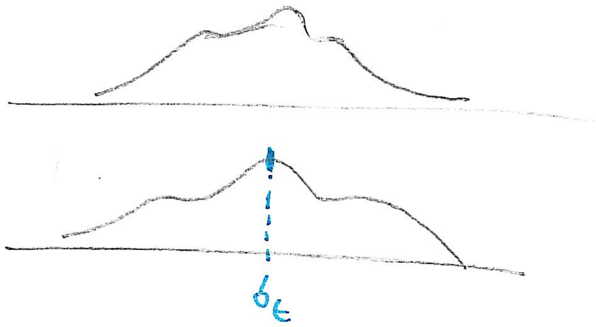
1) Normální apriorno $\mathcal{N}(b,\Sigma)$

$$p(\beta|x_{0:t}, y_{0:t}) \propto f(y_t|x_t,\beta)\cdot p(\beta|b_{t-1},\Sigma_{t-1})$$

2) Najdeme modus = maximum (MAP odhad)

↳ Newtonova iterační metoda, ... atd.

↳ $\mathcal{N}(b_t, ?)$



3) Aproximujeme kovarianci

↳ $\Sigma_t = [\Sigma_{t-1}^{-1} + \hat{y}_t(1-\hat{y}_t)x_t x_t^T]^{-1}$ kde $\hat{y}_t = E[y_t|x_t,\beta]$

⇒ Máme aproximované aposteriorno $\mathcal{N}(b_t,\Sigma_t)$, můžeme znovu updatovat