

Introduction

Algebra and MATLAB review

Mathematical Tools for ITS (11MAI)

Mathematical tools, 2021

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11MAI, lecture 1

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Department of Applied Mathematics, CTU FTS

Introduction

Course Content

Topics to Review

Factoring Polynomials

Taylor Series

MATLAB project

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Website: <http://zolotarev.fd.cvut.cz/mni/>

Lectures and labs take part in all the weeks of the semester, meaning that there is not fixed programme for the last 2–3 weeks.

Course materials only in English, lectures in English as well as in Czech.

Distance learning

See the website for actual distance learning info.

Total of 30 (+5) points:

- 7 assignments (homeworks) \times 3 points each ... 21 points total
- individual semestral project ... 14 points

Minima: 9 points from homeworks, 4 from the project, 16 total

Assigned at the end of most of the labs:

- solved **individually**
- uploaded as PDF to the webserver
- solutions will be typeset, graphs will be **vectors, not bitmaps**, code will be documented
- a set of solutions that are identical will be graded as a single solution, and a corresponding fraction of points will be awarded to every submission in the set

Demonstrates the ability to use knowledge gained during the course. Consists of collected dataset, written report and corresponding MATLAB code.

The report will be typeset, graphs will be **vectors**, code will be documented.

Project topic has to be approved by the instructor by November 30. After that date you will be assigned a topic at the discretion of the instructor.

Evaluation criteria:

- 40 % formal quality of the written report (structuring, citations, etc.)
- 60 % clarity of the presentation
- scaled by the completeness factor $f \in [0, 1]$

Introduction

Course Content

Topics to Review

Factoring Polynomials

Taylor Series

MATLAB project

- a) Polynomials, Taylor polynomials, vector spaces, signals, and images
- b) Series and Fourier series, Discrete Fourier Transform
- c) Stationary and non-stationary signals, windowing and localization
- d) Short Time Fourier Transform, analysis of a non-stationary signal
- e) Data processing review: linear and logistic regression, PCA, clustering
- f) Introduction to numerical computing
- g) Approximation and interpolation, numerical integration of ODEs
- h) Numerical solution to traffic flow PDEs

Knowledge of the following is expected. Use the first lecture and lab to refresh your knowledge.

- Linear algebra, matrix and vector operations
- Derivatives and integration of functions
- Solution of ordinary differential equations
- Series, convergence, Taylor series
- Fundamentals of data processing: linear and logistic regression, regressor selection, regularization, logistic classification, discriminant analysis, principal component analysis, clustering
- Good command of Matlab

In the second lab, an **obligatory** review test takes place. I need **you** to know what shape you are in before we approach some trickier parts of the syllabus.

- Consists of theoretical (algebra, calculus) part and practical (MATLAB) part
- No minimum, only indicative
- Does not contribute to the grading

- Howell K.B.: *Principles of Fourier Analysis*, Chapman & Hall/CRC, 2001
- Smith S.W.: *The Scientist and Engineer's Guide to Digital Signal Processing*, California Technical Publishing
- Broughton S.A., Bryan K.: *Discrete Fourier Analysis and Wavelets*, John Wiley & Sons, 2009
- James G., Witten D., Hastie T., Tibshirani R.: *An introduction to statistical learning*. Springer, 2013.
- Friedman J., Hastie T., Tibshirani, R.: *The elements of statistical learning*. 2nd ed., Springer, 2009.

- MATLAB ® *The Language of Technical Computing*, Getting Started with MATLAB, MathWorks, Inc.
- MATLAB ® *Signal Processing Toolbox User's Guide*, MathWorks, Inc.
- Heath M.T.: *Scientific Computing: An Introductory Survey*. SIAM, 2018.

Introduction

Course Content

Topics to Review

Factoring Polynomials

Taylor Series

MATLAB project

Algebra:

- vectors and matrices, vector and matrix calculus
- linear independence, basis, normality, orthonormality
- linear spaces and subspaces

Calculus:

- derivatives and antiderivatives of x^n , e^{ax} , $\sin x$, $\cos x$
- calculus of composite functions (per-partes, l'Hospital rule, etc.)
- calculus of complex numbers
- solving ODE of up to second order

Matlab requirements:

- working with (normal and cell) vectors and matrices
- basic matrix and vector generating functions
- boolean indexing, `find()`, `mean()`, ...
- plotting, subplots, annotating and saving figures
- reading and saving `.mat`, CSV and sound files
- M-files, and
 - loops (i.e. `for`, `while`)
 - conditionals (i.e. `if`, `else`)

Introduction

Course Content

Topics to Review

Factoring Polynomials

Taylor Series

MATLAB project

Theorem (Fundamental Theorem of Algebra)

Every n th-order polynomial possesses exactly n complex roots

This is a very powerful algebraic tool. It says that given any polynomial

$$\begin{aligned} P_n(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \\ &\equiv \sum_{i=0}^n a_i x^i, \end{aligned}$$

we can always rewrite it as

$$\begin{aligned} P_n(x) &= a_n(x - x_n)(x - x_{n-1}) \cdots (x - x_2)(x - x_1) \\ &\equiv a_n \prod_{i=1}^n (x - x_i) \end{aligned}$$

where the points x_i are the polynomial **roots** and they may be real or complex.

Example (Roots)

Consider the second-order polynomial

$$P_2(x) = x^2 + 7x + 12.$$

The polynomial is second-order because the highest power of x is 2 and is also **monic** because its leading coefficient of x^2 , is $a_2 = 1$.

By the fundamental theorem of algebra there are exactly two roots x_1 and x_2 , and we can write

$$P_2(x) = (x - x_1)(x - x_2).$$

Show that the roots are $x_1 = -3$ and $x_2 = -4$.

The factored form of this simple example is

$$P_2(x) = x^2 + 7x + 12 = (x - x_1)(x - x_2) = (x + 3)(x + 4).$$

Note

Polynomial factorization rewrites a **monic** n -th order polynomial as the product of n first-order monic polynomials, each of which contributes one root (zero) to the product.

This factoring process is often used when working in digital signal processing (DSP).

Factoring can be also performed by MATLAB commands

```
p2 = [1 7 12]; % Polynomial given by its coefficients  
roots(p2) % Print out the roots
```

Example

Find the factors of following polynomials:

- $P_3(x) = x^3 + 2x^2 + 2x + 1$
- $P_2(x) = 9x^2 + a^2$
- $P_4(x) = x^4 - 1$

In order to study the roots of $P_4(x) = x^4 - 1$ using MATLAB, you can write a command creating the polynomial

```
p4 = [1 0 0 0 -1];
```

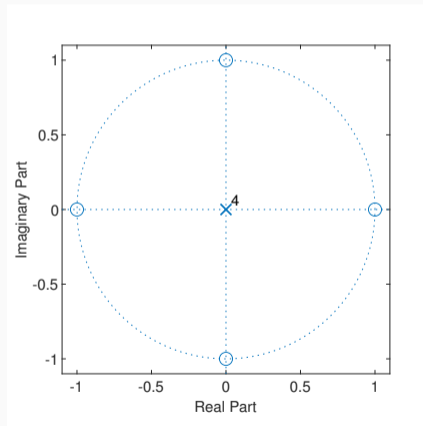
followed by commands

```
roots(p4)
```

to list the roots, or

```
zplane(p4)
```

which gives you a plot of the roots in the complex domain.



Introduction

Course Content

Topics to Review

Factoring Polynomials

Taylor Series

MATLAB project

A Taylor series is a series expansion of a function about a point.

Definition (Taylor series)

A one-dimensional Taylor series is an expansion of a real function $f(x)$, which is $(n + 1)$ -times differentiable, about a point $x = a$ is given by

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + R_n(x)$$

where

$$R_n(x) = \frac{1}{(n + 1)!} \int_a^x f^{(n+1)}(a)(x - a)^{n+1}.$$

The last term $R_n(x)$ is called the **remainder**, or **error term**.

Definition (Taylor polynomial of order n)

A Taylor polynomial of order n is a *partial sum of a Taylor series* (no reminder!):

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3.$$

Taylor polynomials are **local approximations of a function**, which become generally better as n increases.

If $a = 0$, the expansion is also known as a Maclaurin series

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3.$$

Example (Terms of Taylor series)

Evaluate the first five Taylor series terms of $f(x) = 1/(1 - x)$.

$$f(x) = \frac{1}{1 - x} \qquad f(0) = 1$$

$$f'(x) = \frac{1}{(1 - x)^2} \qquad f'(0) = 1$$

$$f''(x) = \frac{2}{(1 - x)^3} \qquad f''(0) = 2$$

$$f'''(x) = \frac{2 \times 3}{(1 - x)^4} \qquad f'''(0) = 6$$

$$f^{(4)}(x) = \frac{6 \times 4}{(1 - x)^5} \qquad f^{(4)}(0) = 24$$

Example (Terms of Taylor series continued)

And as

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

we have

$$\frac{1}{1-x} \approx 1 + x + x^2 + x^3 + x^4.$$

Do you remember the formula for the sum of geometric series?

... e (Euler's number) and $\sqrt{-1}$.

$$i \equiv \sqrt{-1}$$

$$e \equiv \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = 2.71828182845905 \dots$$

The first, $i = \sqrt{-1}$, is the basis for complex numbers, called **imaginary unit**.

The second, $e = 2.718 \dots$, is a **transcendental** real number defined by the above limit. It is the base of the natural logarithm.

Let us work a bit further on polynomial expansion.

Example (Harmonic and exponential approximations)

Use Taylor expansion to find approximations of the following functions up to 3 terms:

- $f(x) = e^x$
- $f(x) = \sin x$
- $f(x) = \cos x$

Example (Harmonic and exponential approximations)

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

If we introduce **imaginary unit** ix in the exponential approximation above we obtain

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \dots$$

For imaginary unit we have

$$i^1 = \sqrt{-1}$$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

so the exponential approximation takes the form

$$e^{ix} = 1 + i\frac{x}{1!} - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

It can be regrouped and easily identified with

$$e^{ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + i\frac{x}{1!} - i\frac{x^3}{3!} + \cdots \equiv \cos x + i \sin x$$

Definition (Euler's formula)

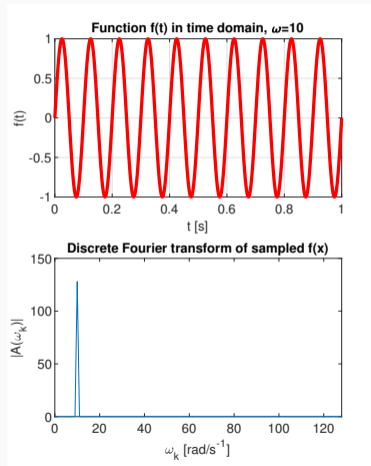
The result is the famous Euler's formula (1743 Opera Omnia, vol. 14, p. 142)

$$e^{ix} = \cos x + i \sin x$$

Euler's identity is the key to understanding the meaning of expressions like

$$f(\omega_k T) \equiv e^{i\omega_k T} = \cos(\omega_k T) + i \sin(\omega_k T).$$

We will see later that such an expression defines a sampled complex sinusoid.





Introduction

Course Content

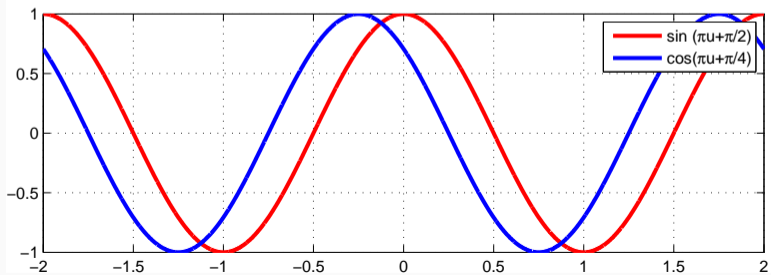
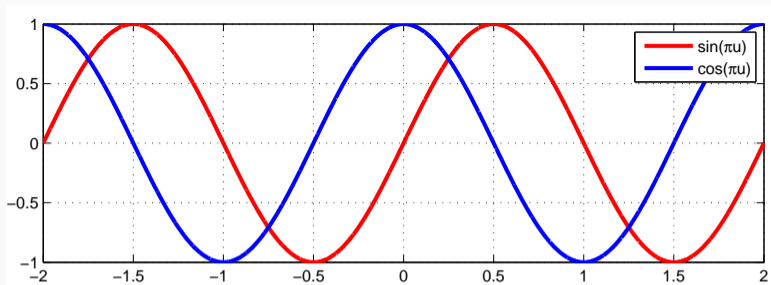
Topics to Review

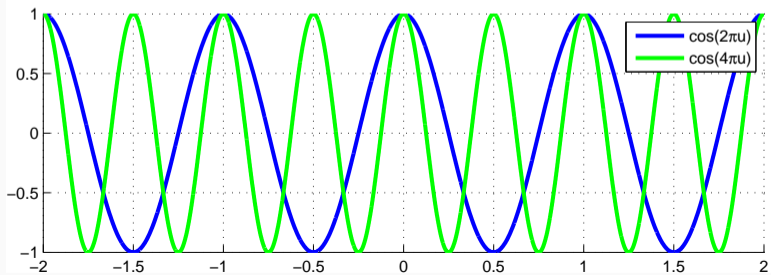
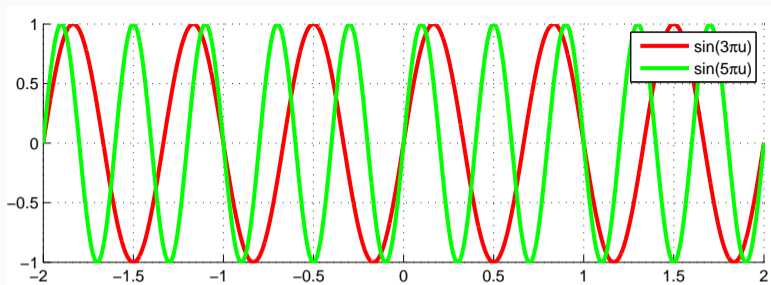
Factoring Polynomials

Taylor Series

MATLAB project

- a) Using MATLAB plot the graphs of the sine and cosine functions, $\sin(\pi u)$ and $\cos(\pi u)$ within the interval $-2 \leq u \leq 2$.
- b) Plot graphs of the functions $\sin(\pi u + \pi/2)$ and $\cos(\pi u + \pi/4)$ within the interval $-2 \leq u \leq 2$.
- c) Plot graphs of the functions $\sin(3\pi u)$ and $\sin(5\pi u)$ within the interval $-2 \leq u \leq 2$.
- d) Display axes, add legends to all graphs.
- e) Save every output as a PNG, EPS, and Windows EMF file.





- a) Derive the formulae for factoring the following polynomials for arbitrary $n \in \mathbb{N}$:

$$P_{2n}(x) = x^{2n} \pm 1$$

$$P_{2n+1}(x) = x^{2n+1} \pm 1$$

- b) Check your results using MATLAB library function `roots()` for finding the roots of a polynomial.
- c) Plot the roots of polynomials of degree $2n = 16$ and $2n = 32$ using MATLAB library function `zplane()`.
- d) Follow the symmetrical properties of the roots. Report on what do you observe.
- e) Deliver your results by Friday, October 8 2021 to the web page <http://zolotarev.fd.cvut.cz/mni>.

Solution report should be formally correct (structuring, grammar).

Only **.pdf** files are acceptable. Handwritten solutions and **.doc** and **.docx** files will not be accepted.

Solutions written in $\text{T}_\text{E}\text{X}$ (using LyX, Overleaf, whatever) may receive small bonification.