Vector space of continuous periodic functions
Fourier series

Mathematical Tools for ITS (11MAI)

Mathematical tools, 2021

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Home work

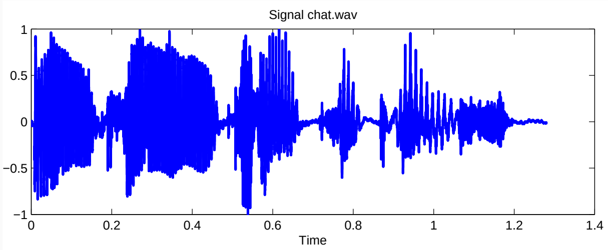
We recognize fundamentally 1-dimensional, 2-dimensional, and multidimensional signals.

1D

a) a real piano tone A 

b) a speech 

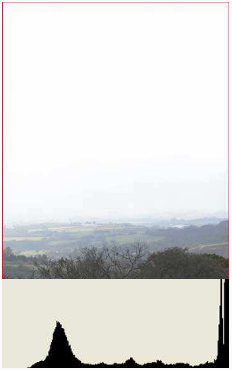
1-dimensional signals



2-dimensional signals

2-dimensional signals

Histogram is a graph showing the number of pixels in an image at each different intensity value.



Common Image Processing Problems

- Image restoration and denoising
- Edge detection and denoising
- Image compression

Image denoising by filter application

Image restoration and denoising

Edge detection and denoising

Edge detection and denoising

Restoration, image denoising

Images can be of poor quality for variety reasons:

- low-quality image capture (security video cameras)
- blurring when the picture is taken
- physical damage to an actual photo
- noise contamination during the image capture process

Restoration seeks to return the image to its original quality.

The features of interest in an image are the edges, areas of transition that indicate the end of one object and beginning of another.

Applicable in image processing — see Lena[1], or in automated vision and robotics.

[1] Lena Soderberg (*Sjööblom*) 1972, `https://en.wikipedia.org/wiki/Lena_Fors%C3%A9n`

Memory requirement for a typical photograph:

- 24-bit colour $\equiv$ 1 byte for each of the red, green, and blue components
- for 2048×1526 pixel image we need $2048 \times 1526 \times 3 = 9431040$ bytes
- 9 MB a picture, what can be stored in a 2 GB memory stick ?

Compression algorithms !!! and their drawbacks

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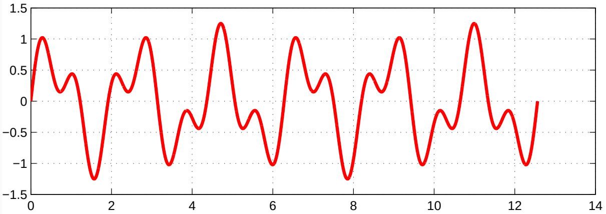
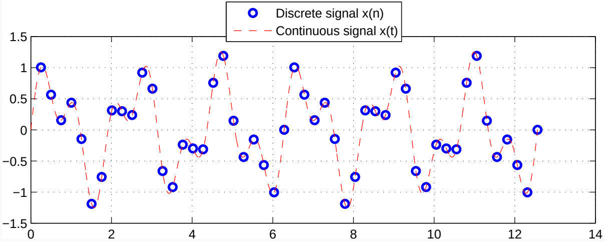
Home work

- An analog or continuous signal $x(t)$ is a real-valued function of an independent variable t in the definition domain $a \leq t \leq b$; variable t is usually time.
- The function $x(t)$ can represent the intensity of a sound (audio signal), the speed of an object, ...
- For $N \geq 1$ we define the sampling period $T_s = \dfrac{b-a}{N}$, the quantity $f_s = \dfrac{1}{T_s}$ is proportional to number of samples taken during each time period and it is called sampling frequency.
- Finally we obtain digital or sampled signal $x(n) = x(a + n \times T_s)$ for $0 \leq n \leq N$.


1.5
1
0.5
0
−0.5
−1
−1.5
0
2
4
6
8
10
12
14

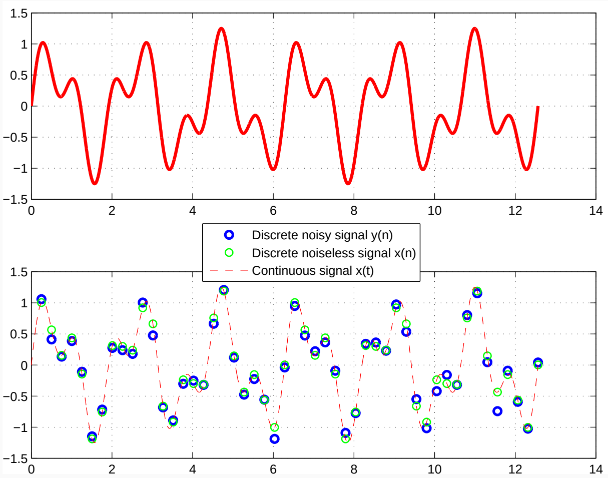

Discrete signal x(n)
Continuous signal x(t)
1.5
1
0.5
0
−0.5
−1
−1.5
0
2
4
6
8
10
12
14

- Sampling is not the only source of error in A/D conversion.
- Consider an analog voltage signal that ranges from 0 to 1 volt.
- An A-to-D converter divides up this 1 volt range into $2^8 = 256$ equally sized intervals.
- The k-th voltage interval is given by $k\Delta u \le u < (k+1)\Delta u$ where $\Delta u = 1/256\,\mathrm{V}$ and $k \in \mathbb{N}_0, 0 \le k \le 255$.
- If a measurement of the analog signal at an instant in time falls within the k-th interval, then the A-to-D converter records the voltage at this time as $k\Delta u$.

- This $k\Delta u$ is the quantization step, in which a continuously varying quantity is truncated or rounded to the nearest of a finite set of values $\Rightarrow$ quantization error.
- An A-to-D converter as above would be said to be 8-bit, because each analog measurement is converted into an 8-bit quantity.

The combination of sampling and quantization allows us to digitize a continous signal or image, and thereby convert it into a form suitable for computer storage and processing.

Quantization Error $y(n) = x(n) + \epsilon(n)$



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Definition (Vector space)

A vector space over the real numbers $\mathbb{R}$ is a set $\mathcal{V}$ with two operations, vector addition and scalar multiplication, with the properties that

a) for all vectors $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ the vector sum $\mathbf{u} + \mathbf{v}$ is defined and the result lies again in $\mathcal{V}$ (closure under addition);
b) for all $\mathbf{u} \in \mathcal{V}$ and scalars $a \in \mathbb{R}$ the scalar multiple $a \cdot \mathbf{u}$ is defined and lies in $\mathcal{V}$ (closure under scalar multiplication);
c) the familiar rules of arithmetic apply

If we replace $\mathbb{R}$ above by the field of complex numbers $\mathbb{C}$, then we obtain the definition of a vector space over the complex numbers.

Specifically, for all scalars $a, b \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}$:

1) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$, e.g. addition is commutative,
2) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ e.g. addition is associative,
3) there is a zero vector $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} \equiv \mathbf{u}$ (additive identity),
4) for each $\mathbf{u} \in V$ there is an additive inverse vector $\mathbf{w}$ such that $\mathbf{u} + \mathbf{w} = \mathbf{0}$, we conventionally write $-\mathbf{u}$ for the additive inverse of $\mathbf{u}$,
5) $(ab)\mathbf{u} = a(b\mathbf{u})$,
6) $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$,
7) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.

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Example

The vector space $\mathbb{R}^N$ consists of vectors $\mathbf{x}$ of the form $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ where the x_k are all real numbers. Prove all the properties of the vector space, e.g. multiplication, addition ...

Warning: In later work we will almost always find it convenient to index the components of vectors in $\mathbb{R}^N$ or $\mathbb{C}^N$ starting with index 0, that is, as $\mathbf{x} = (x_0, x_1, ..., x_{N-1})$, rather than the more traditional range 1 to N.

Example

The sets $\mathbb{M}_{m,n}(\mathbb{R})$ or $\mathbb{M}_{m,n}(\mathbb{C})$, $m \times n$ matrices with real or complex entries respectively, form vector spaces.

Note:

Any multiplicative properties of matrices are irrelevant in this context!!

The vector space $\mathbb{M}_{m,n}(\mathbb{R})$ is an appropriate model for the discretization of images on a rectangle. Analysis of images is often facilitated by viewing them as members of space $\mathbb{M}_{m,n}(\mathbb{C})$.

Vectors in $\mathcal{V}$ can be (i) multiplies by scalars, (ii) added. Using both operation at ones leads to linear combination of vectors.

Definition (Linear Combination)

A vector $\mathbf{v}$ in vector space $\mathcal{V}$ is a linear combination of vectors $\mathbf{u}_1; \mathbf{u}_2; \ldots; \mathbf{u}_m \in \mathcal{V}$ if there exist scalars $a_1; a_2; \ldots; a_m$ such that

$$\mathbf{v} = a_1 \cdot \mathbf{u}_1 + a_2 \cdot \mathbf{u}_2 + \cdots + a_m \cdot \mathbf{u}_m.$$

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Definition (Basis)

A set $\mathcal{B}$ of elements (vectors) in a vector space $\mathcal{V}$ is called a basis, if every element of $\mathcal{V}$ can be written in a unique way as a linear combination of elements of $\mathcal{B}$.

Recall that

- this implies that all basis vectors are linearly independent,
- the coefficients of the linear combination are coordinates of the vector w.r.t. basis $\mathcal{B}$,
- the dimension of $\mathcal{V}$ is given by cardinality of $\mathcal{B}$,
- there is one and only one way to write $\mathbf{v} \in \mathcal{V}$ as a combination of the basis vectors.

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Recall that each vector $\mathbf{u}$ in n-dimensional space $\mathbb{R}^n$ can be uniquely represented as a linear combination of n basis vectors $\mathbf{e_1}, \ldots, \mathbf{e_n}$:

$$\mathbf{u} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \cdots + \alpha_n \mathbf{e}_n = \sum_{i=0}^{N} \alpha_i \mathbf{e}_i,$$

How do we compute the coordinates, i.e. the values of $\alpha_i \in \mathbb{R}$?

The traditional approach is to solve a set of linear equations for particular elements of $\mathbf{u} = (u_1, u_2, \ldots, u_n)^\top$, but this is quite demanding . . .

Luckily for us, there is a better way.

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Definition (Inner product)

Operation that assigns a non-negative scalar to a pair of vectors $\mathbf{u}$ and $\mathbf{v}$, denoted $\langle \mathbf{u}, \mathbf{v} \rangle$, is called an inner product on $\mathcal{V}$ if it satisfies the following:

a) $\langle a \cdot \mathbf{u} + b \cdot \mathbf{w}, \mathbf{v} \rangle = a \cdot \langle \mathbf{u}, \mathbf{v} \rangle + b \cdot \langle \mathbf{w}, \mathbf{v} \rangle$

b) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

c) $\langle \mathbf{u}, \mathbf{v} \rangle \geq 0$, and $\langle \mathbf{u}, \mathbf{u} \rangle = 0 \iff \mathbf{v} \equiv \mathbf{0}$

As $\langle \mathbf{u}, \mathbf{v} \rangle \geq 0$, we also have the following:

Definition (Norm of a vector)

The norm or length of a vector $\mathbf{u} \in \mathcal{V}$ is given by

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} \qquad \|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$$

Definition (Inner product space)

Inner product space is a vector space with inner product operation defined.

For inner product space we still have $\mathbf{u} \in \mathcal{V}$ as a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$

$$\mathbf{u} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \cdots + \alpha_n \mathbf{e}_n,$$

but in addition $\alpha_i \in \mathbb{R}$ can be computed using the inner product $\langle \cdot, \cdot \rangle$ as

$$\alpha_i = \langle \mathbf{u}, \mathbf{e}_i \rangle$$

Example

Starting with $\langle \mathbf{u}, \mathbf{e}_i \rangle = \langle \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \cdots + \alpha_n \mathbf{e}_n, \mathbf{e}_i \rangle$ show that indeed $\alpha_i = \langle \mathbf{u}, \mathbf{e}_i \rangle$.

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Definition (Orthornormal vectors)

Vectors $\mathbf{e}_i$ are orthonormal if they are

a) orthogonal, i.e. it holds that $\forall i \neq j : \langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0$

b) normalized, i.e. it holds that $\forall i : \langle \mathbf{e}_i, \mathbf{e}_i \rangle = \|\mathbf{e}_i\|^2 = 1$

Example

Draw the following two vectors and their sum in the two-dimensional Euclidean space $\mathbb{R}^2$

$$\mathbf{u} = 3 \cdot \mathbf{e}_1 + 4 \cdot \mathbf{e}_2$$

$$\mathbf{v} = -2 \cdot \mathbf{e}_1 + 3 \cdot \mathbf{e}_2$$

and make them normalized.

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Vectors are objects that can be added together and multiplied by scalars – vector space:

If $\mathbf{u} = \sum_{i=1}^{n} \alpha_i \mathbf{e}_i$ and $\mathbf{v} = \sum_{i=1}^{n} \beta_i \mathbf{e}_i$ and λ is some scalar, then

$$\mathbf{u} + \mathbf{v} = \sum_{i=1}^{n} (\alpha_i + \beta_i)\mathbf{e}_i$$

$$\lambda \mathbf{u} = \sum_{i=1}^{n} \lambda \alpha_i \mathbf{e}_i$$

Vector space of continuous-time signals

We have already studied the space of continuous-time signals. We can easily verify:

- we can form the sum of any two signals $x_1(t)$ and $x_2(t)$ to obtain another signal

$$x(t) = x_1(t) + x_2(t)$$

- we can multiply any signal $x(t)$ by a constant λ to obtain another signal

$$y(t) = \lambda x(t)$$

Unlike the n-dimensional space $\mathbb{R}^n$, the vector space of all continuous-time signals is infinite-dimensional.

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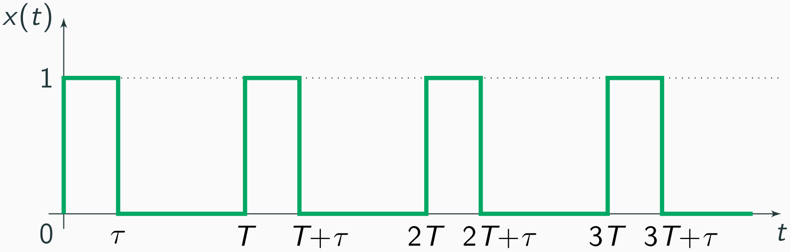
Periodic signals

Definition

Any signal $x(t)$ that satisfies the periodicity condition

$$\forall t : x(t+T) = x(t)$$

for given period T is called periodic signal with period T.



Vector space of periodic signals

It is easy to see that periodic signals form a vector space:

- if $x_1(t)$ and $x_2(t)$ are periodic, then

  $$x(t+T) = x_1(t+T) + x_2(t+T) = x_1(t) + x_2(t) = x(t)$$

  is also periodic with the same period T
- if $x_1(t)$ is periodic and λ is scalar, then

  $$y(t+T) = \lambda x(t+T) = \lambda x(t) = y(t)$$

  is a scaled version of $x(t)$ being also periodic with period T

If we impose even more conditions on periodic signals – the Dirichlet conditions, which hold for all signals encountered in practice, then we can represent signals as infinite linear combinations of orthogonal and normalized vectors.

- A function satisfying Dirichlet conditions must have right and left limits at each point of discontinuity:

$$x(t+) = \lim_{\tau \to t+} x(\tau) \qquad \text{and} \qquad x(t-) = \lim_{\tau \to t-} x(\tau)$$

- The Dirichlet theorem says in particular that the Fourier series for $x(t)$ converges and is equal to

$$x(t) = \frac{x(t+) + x(t-)}{2}$$

  wherever $x(t)$ is continuous.

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Definition (Inner product of T-periodic signals)

We can define the inner product of two T-periodic signals $x_1(t)$ and $x_2(t)$ as

$$\langle x_1(t), x_2(t) \rangle = \int_0^T x_1(t) x_2(t) \mathrm{d}t.$$

As the signal is periodic, we can integrate over any complete period, i.e. from $-T/2$ to $T/2$ or from $-T$ to 0:

$$\langle x_1(t), x_2(t) \rangle = \int_{-\frac{T}{2}}^{\frac{T}{2}} x_1(t) x_2(t) \mathrm{d}t = \int_{-T}^{0} x_1(t) x_2(t) \mathrm{d}t.$$

Then we can take any sequence of T-periodic functions $\{\phi_j(t)\}_{j\in\mathbb{N}}$ that are

- normalized – $\langle \phi_j(t), \phi_j(t)\rangle = \|\phi_j(t)\|^2 = \int_0^T \phi_j^2(t)\mathrm{d}t = 1$,
- orthogonal – $\langle \phi_j(t), \phi_j(t)\rangle = \int_0^T \phi_j(t)\phi_k(t)\mathrm{d}t = 0$ for $j \neq k$
- complete – if a signal $x(t)$ is such that

$$\langle \phi_j(t), x(t)\rangle = \int_0^T \phi_j(t)x(t)\mathrm{d}t = 0$$

for all j, then $x(t) = 0$.

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Definition (Fourier series)

Let $\{\phi_j(t)\}_j$, $j \in \mathbb{N}$, be a complete orthonormal set of functions. Then any well-behaved T-periodic signal $x(t)$ can be uniquely represented as

$$x(t) = \sum_{j=0}^{\infty} \alpha_j \phi_j(t).$$

This is called the Fourier series representation of $x(t)$.

The scalars

$$\alpha_j = \langle \phi_j(t), x(t) \rangle = \int_0^T \phi_j(t) x(t) \mathrm{d}t.$$

are called the Fourier coefficients of $x(t)$ with respect to ϕ_j.

To derive the formula for α_j, write

$$x(t)\phi_k(t) = \sum_{j=0}^{\infty} \alpha_j \phi_j(t)\phi_k(t)$$

and then integrate over a period, effectively computing an innter product:

$$\langle \phi_k(t), x(t) \rangle = \int_0^T \phi_k(t)x(t)\mathrm{d}t = \int_0^T \sum_{j=0}^{\infty} \alpha_j \phi_j(t)\phi_k(t)\mathrm{d}t.$$

For convergent series we can integrate term by term, hence

$$\langle \phi_k(t), x(t) \rangle = \int_0^T \sum_{j=0}^{\infty} \alpha_j \phi_j(t)\phi_k(t)\mathrm{d}t = \sum_{j=0}^{\infty} \alpha_j \int_0^T \phi_j(t)\phi_k(t)\mathrm{d}t = \sum_{j=0}^{\infty} \alpha_j \delta_{j,k} = \alpha_k$$

Here and in following evaluation we will use Kronecker delta which is defined as $\delta_{j,k} = 0$ for $j \neq k$ and $\delta_{k,k} = 1$ and which indicates that $\{\phi_j(t)\}_{j=0}^{\infty}$ form an orthonormal system of functions.

In analogy to vectors in n-dimensional space, you can think of α_j as the projection of $x(t)$ in the direction of $\phi_j(t)$.

It can be also proved that, as the functions $\{\phi_j(t)\}_{j=0}^{\infty}$ form a complete orthonormal system, the partial sums of the Fourier series

$$x(t) = \sum_{j=0}^{\infty} \alpha_j \phi_j(t)$$

converge to $x(t)$ in the following sense (L_2-convergence):

$$\lim_{N\to\infty} \int_0^T \left(x(t) - \sum_{j=0}^{N} \alpha_j \phi_j(t) \right)^2 \mathrm{d}t = 0.$$

Similarly to the case of Taylor polynomial, we can use (with some care for discontinuities) the partial sum

$$x(t) \approx \sum_{j=0}^{N} \alpha_j \phi_j(t)$$

to approximate $x(t)$.

The approach described so far can be used for arbitrary choice of basis $\{\phi_j(t)\}_{j=0}^{\infty}$ as long as it is

- complete, and
- orthonormal.

We will now review two most frequently used choices of basis:

- Trigonometric basis (i.e. $\cos \omega_k t$, $\sin \omega_k t$),
- Complex exponential basis (i.e. $e^{j\omega_k t}$).

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Definition (Fundamental frequency)

When sampling a signal with sampling period T, the fundamental frequency, ω_0, of the signal is

$$\omega_0 = \frac{2\pi}{T}.$$

Definition (Trigonomeric basis)

The sequence of T-periodic functions $\{\phi_j(t)\}_{j=0}^{\infty}$ defined for $k = 1, 2, \ldots$ and $\omega_k = k\omega_0$ by

$$\text{a)}\ \phi_0(t) = \frac{1}{\sqrt{T}}$$

$$\text{b)}\ \phi_{2k-1}(t) = \sqrt{\frac{2}{T}} \cos \omega_k t$$

$$\text{c)}\ \phi_{2k}(t) = \sqrt{\frac{2}{T}} \sin \omega_k t$$

is complete and orthonormal.

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Note

Note the first few functional elements of the sequence from the previous slide (without scaling factors):

$$\{1, \cos t, \sin t, \cos 2t, \sin 2t, \cos 3t, \sin 3t, \dots\}$$

Trigonometric Fourier Series

Common way of writing down the trigonometric Fourier series of $x(t)$ is following:

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos \omega_k t + \sum_{k=1}^{\infty} b_k \sin \omega_k t$$

The Fourier coefficients can be computed as follows:

a) $a_0 = \frac{1}{T} \int_0^T x(t) \mathrm{d}t$

b) $a_k = \frac{2}{T} \int_0^T x(t) \cos \omega_k t \mathrm{d}t$

c) $b_k = \frac{2}{T} \int_0^T x(t) \sin \omega_k t \mathrm{d}t$

To relate this to the orthonormal representation in terms of the $\{\phi_j(t)\}_{j\in\mathbb{N}}$ and its Fourier coefficients α_j, we note that

a) $a_0 = \frac{1}{T}\int_0^T x(t)\mathrm{d}t = \frac{1}{\sqrt{T}}\int_0^T x(t)\,\frac{1}{\sqrt{T}}\mathrm{d}t = \frac{1}{\sqrt{T}}\int_0^T x(t)\,\phi_0(t)\mathrm{d}t = \frac{1}{\sqrt{T}}\,\alpha_0$

b) $a_k = \cdots$

c) $b_k = \cdots$

Hence,

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos\omega_k t + \sum_{k=1}^{\infty} b_k \sin\omega_k t \equiv \sum_{j=0}^{\infty} \alpha_j\phi_j(t).$$

To relate this to the orthonormal representation in terms of the $\{\phi_j(t)\}_{j\in\mathbb{N}}$ and its Fourier coefficients α_j, we note that

a) $a_0 = \frac{1}{\sqrt{T}}\alpha_0 \quad \Longleftrightarrow \quad \alpha_0 = \sqrt{T}a_0$

b) $a_k = \cdots$

c) $b_k = \cdots$

Hence,

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos\omega_k t + \sum_{k=1}^{\infty} b_k \sin\omega_k t \equiv \sum_{j=0}^{\infty} \alpha_j \phi_j(t).$$

To relate this to the orthonormal representation in terms of the $\{\phi_j(t)\}_{j\in\mathbb{N}}$ and its Fourier coefficients α_j, we note that

a) $a_0 = \frac{1}{\sqrt{T}}\alpha_0 \quad \Longleftrightarrow \quad \alpha_0 = \sqrt{T}a_0$

b) $a_k = \frac{2}{T}\int_0^T x(t)\cos\omega_k t\,\mathrm{d}t = \sqrt{\frac{2}{T}}\int_0^T x(t)\sqrt{\frac{2}{T}}\cos\omega_k t\,\mathrm{d}t = \sqrt{\frac{2}{T}}\int_0^T x(t)\,\phi_{2k-1}(t)\mathrm{d}t$
$= \sqrt{\frac{2}{T}}\,\alpha_{2k-1}$

c) $b_k = \cdots$

Hence,

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k\cos\omega_k t + \sum_{k=1}^{\infty} b_k\sin\omega_k t \equiv \sum_{j=0}^{\infty}\alpha_j\phi_j(t).$$

To relate this to the orthonormal representation in terms of the $\{\phi_j(t)\}_{j\in\mathbb{N}}$ and its Fourier coefficients α_j, we note that

a) $a_0 = \frac{1}{\sqrt{T}}\alpha_0 \iff \alpha_0 = \sqrt{T}a_0$

b) $a_k = \sqrt{\frac{2}{T}}\alpha_{2k-1} \iff \alpha_{2k-1} = \sqrt{\frac{T}{2}}a_k$

c) $b_k = \cdots$

Hence,

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos\omega_k t + \sum_{k=1}^{\infty} b_k \sin\omega_k t \equiv \sum_{j=0}^{\infty} \alpha_j \phi_j(t).$$

To relate this to the orthonormal representation in terms of the $\{\phi_j(t)\}_{j\in\mathbb{N}}$ and its Fourier coefficients α_j, we note that

a) $a_0 = \frac{1}{\sqrt{T}}\alpha_0 \quad \Longleftrightarrow \quad \alpha_0 = \sqrt{T}a_0$

b) $a_k = \sqrt{\frac{2}{T}}\alpha_{2k-1} \quad \Longleftrightarrow \quad \alpha_{2k-1} = \sqrt{\frac{T}{2}}a_k$

c) $b_k = \frac{2}{T}\int_0^T x(t)\sin\omega_k t\mathrm{d}t = \sqrt{\frac{2}{T}}\int_0^T x(t)\sqrt{\frac{2}{T}}\sin\omega_k t\mathrm{d}t = \sqrt{\frac{2}{T}}\int_0^T x(t)\,\phi_{2k}(t)\mathrm{d}t$
$= \sqrt{\frac{2}{T}}\alpha_{2k}$

Hence,

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k\cos\omega_k t + \sum_{k=1}^{\infty} b_k\sin\omega_k t \equiv \sum_{j=0}^{\infty}\alpha_j\phi_j(t).$$

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Even symmetry nulls sine coefficients

If $x(t)$ is an even function, i.e., $x(t) = x(-t)$ for all t, then all its sine Fourier coefficients are zero:

$$b_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\sin\omega_k t\,\mathrm{d}t = 0$$

Odd symmetry nulls cosine coefficients

If $x(t)$ is an odd function, i.e., $x(t) = -x(-t)$ for all t, then all its cosine Fourier coefficients are zero:

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\cos\omega_k t\,\mathrm{d}t = 0$$

Theorem (Fourier series of an even function)

Fourier series of an even function $f(t) = f(-t)$ consists of the constant and cosine terms

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t),$$

where $\omega_0 = \dfrac{2\pi}{T}$.

Theorem (Fourier series of an odd function)

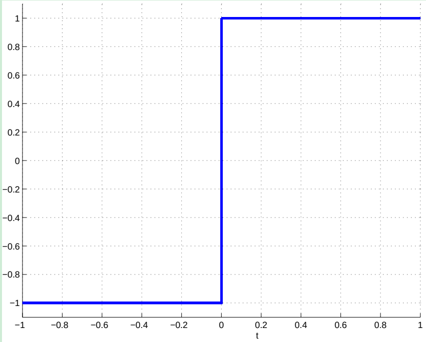
Fourier series of an odd function $f(t) = -f(-t)$ consists of the sine terms

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t),$$

where $\omega_0 = \dfrac{2\pi}{T}$.

Example (Square wave)

Find the trigonometric Fourier series representation of a periodic signal $x(t) = x(t+T)$ given by repeating the square wave



Note, that in this case $T = 2$.

Solution:

a) the signal has odd symmetry $\Rightarrow$ all $a_k = 0$

b) $b_k = \dfrac{2}{T}\displaystyle\int_{-1}^{1} x(t)\sin\omega_k t\mathrm{d}t = \dfrac{2}{T}\int_{-1}^{0}(-1)\sin\omega_k t\mathrm{d}t + \dfrac{2}{T}\int_{0}^{1}(+1)\sin\omega_k t\mathrm{d}t$

$$= \frac{2}{T}\left[\frac{\cos\omega_k t}{\omega_k}\right]_{-1}^{0} - \frac{2}{T}\left[\frac{\cos\omega_k t}{\omega_k}\right]_{0}^{1} = \frac{1}{k\pi}\left[\cos\omega_k t\right]_{-1}^{0} - \frac{1}{k\pi}\left[\cos\omega_k t\right]_{0}^{1}$$

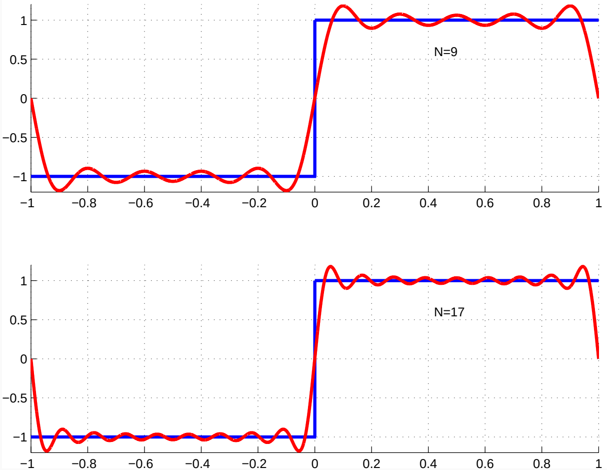
$$= \frac{2}{k\pi}(1-\cos(k\pi)) = \frac{4}{k\pi}\sin^2(\frac{k\pi}{2})$$

c) For $k = 2m-1$ is $b_k = \dfrac{4}{k\pi}\sin^2(\dfrac{k\pi}{2}) = \dfrac{4}{(2m-1)\pi}$

d) $x(t) = \displaystyle\sum_{m=1}^{\infty}\frac{4}{(2m-1)\pi}\sin((2m-1)\pi t)$

Partial sums

$$x_N(t) = \sum_{m=1}^{N} \frac{4}{(2m-1)\pi} \sin(2m-1)\,\pi t$$



Gibbs phenomenon

The Fourier series (over/under)shoots the actual value of $x(t)$ at points of discontinuity regardless of degree N.

Complex exponentials

Another useful complete orthonormal set is accomplished by the complex exponentials:

$$\phi_k(t) = \frac{1}{\sqrt{T}} \mathrm{e}^{jk\omega_0 t}, k \in \mathbb{Z}$$

These functions are complex-valued! We have to evaluate the inner product as

$$\langle x_1(t), x_2(t) \rangle = \int_0^T x_1(t) x_2^*(t) \mathrm{d}t,$$

where $x_2^*(t)$ denotes complex conjugation.

Complex exponential Fourier series

a) $\langle \phi_k(t), \phi_\ell(t) \rangle = \dfrac{1}{T} \displaystyle\int_0^T \mathrm{e}^{j\omega_k t} \cdot \mathrm{e}^{-j\omega_\ell t} \mathrm{d}t = \delta_{k,\ell}$

b) $x(t) = \displaystyle\sum_{k=-\infty}^{\infty} c_k \, \mathrm{e}^{j\omega_k t}$

c) $c_k = \dfrac{1}{T} \displaystyle\int_0^T x(t) \, \mathrm{e}^{-j\omega_k t} \mathrm{d}t$

d) as in trigonometric case $\omega_0 = \dfrac{2\pi}{T}$, $\omega_k = k\omega_0$, $\omega_\ell = \ell\omega_0$.

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Vector space of periodic signals

Complete orthonormal systems of functions

Trigonometric and complex exponential Fourier Series

Matlab project

Home work

Matlab Project 2.1 — Sampling a sine function (1/2)

a) Start Matlab and open a new M-file.

b) Consider sampling the function

$$f(t) = \sin(2\pi(440)t)$$

on the interval $0 \leq t < 1$, at 8192 points

Hint: Choose sampling interval $\Delta t = 1/8192$ to obtain samples $f(k) = f(k\Delta t) = \sin(2\pi(440)k/8192)$ for $0 \leq k \leq 8191$. The samples can be arranged in a vector `f`; you can do this in Matlab with

```
f = sin(2*pi * 440 * (0:8191)/8192);
```

Note

The sample vector `f` is stored in double precision floating point, about 15 significant digits. However, we'll consider `f` as not yet quantized. That is, the individual components `f(k)` of `f` can be thought of as real numbers that vary continuously, since 15 digits is pretty close to continuous for our purposes.

Qestions and tasks:

1) What is the frequency in Hertz of the harmonic function $f(t)$?
2) Plot the sampled signal with the command `plot(f)`. It probably doesn't look too good, as it goes up and down 440 times in the plot range. Plot a smaller range, say the first 100 samples.
3) At the sampling rate 8192 Hz, what is the Nyquist frequency? Is the frequency of $f(t)$ above or below the Nyquist frequency?
4) Type `sound(f)` to play the sound out of the computer speaker.
   Note: By default, Matlab plays all sound files at 8192 samples per second, and assumes the sampled audio signal is in the range -1 to 1.

Matlab Project 2.2 – Aliasing and quantization (1/4)

Consider a second signal

$$g(t) = \sin(2\pi(8632)t) = \sin(2\pi(440 + 8192)t).$$

Repeat parts (a) through (d) from previous part with sampled signal

```
g = sin(2*pi * (440+8192) * (0:8191)/8192);
```

The analog signal $g(t)$ oscillates much faster than $f(t)$, and we could expect it to yield a higher pitch. However, when sampled with $f_s = 8192$ Hz, $f(t)$ and $g(t)$ are aliased and yield precisely the same sampled vectors `f` and `g`. They should sound the same too.

Matlab Project 2.2 – Aliasing and quantization (2/4)

Qestions and tasks:

1) To illustrate the effect of quantization error, construct a 2-bit (4 quantization levels) version of the audio signal $f(t)$. The command

```
qf2 = ceil(2 * (f+1))-1;
```

applied to `f` produces the quantized signal `qf2`.

Sample values of $f(t)$ in the ranges $(-1, -0.5]$, $(-0.5, 0]$, $(0, 0.5]$, and $(0.5, 1]$ are mapped to the integers $0, 1, 2, 3$, respectively.

Note that the value -1 will be mapped to -1. Look into `find()` method or logical indexing for approaches how to replace all -1 values in `qf2` with zeros.

b) To approximately reconstruct the quantized signal, apply the dequantization formula to reconstruct `f` as `fr2` using

```
fr2 = -1 + 0.5 * (qf2+0.5);
```

This maps the integers $0, 1, 2$ and 3 to values $-0.75, -0.25, 0.25$, and 0.75, respectively.

c) Plot the first hundred values of `fr2` with `plot(fr2(1:100))`.

d) Play the quantized signal with `sound(fr2)`.

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e) Compute the distortion (as a percentage) between the sampled signal vector $\mathbf{f}$ and the dequantized signal vector $\mathbf{f}_r$ using

$$\epsilon = \frac{\|\mathbf{f} - \mathbf{f}_r\|^2}{\|\mathbf{f}\|^2}$$

The command `norm(f)` command in MATLAB computes the standard Euclidean norm of the vector $\|\mathbf{f}\|^2$.

Consider the half-wave rectified sinusoid function,

$$f(t) = \begin{cases} \sin\left(\dfrac{2\pi t}{T}\right) & \text{if} \quad 0 \leq t \leq \dfrac{T}{2}, \\ 0 & \text{if} \quad \dfrac{T}{2} \leq t \leq T. \end{cases}$$

1) Find the trigonometric Fourier series representation of $f(t)$.
   Hint: Calculate the coefficients a_k and b_k using identities

$$2\sin \ell x \sin mx = \cos(\ell - m)x - \cos(\ell + m)x,$$
$$2\sin \ell x \cos mx = \sin(\ell - m)x + \sin(\ell + m)x.$$

2) Plot the first 5 components of the Fourier series using Matlab.

Consider now the sawtooth function

$$f(t) = f(t+T) = t, \qquad -\frac{T}{2} \leq t \leq \frac{T}{2}.$$

1) As the function $f(t)$ is odd, the coefficients $a_k = 0$. Calculate coefficients b_k.
2) Plot the first 5 components of the Fourier series using Matlab.

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Assignment 2.1 — Matlab experiment

a) Repeat the quantization process from Project 2.2 using 4-,5-,6-, and 8-bit quantization.

Example: 5-bit quantization is accomplished with

```
qf5 = ceil(16*(f+1))-1
```

and dequantization with

```
fr5 = -1 + (qf+0.5)/16
```

Hint for other quantization levels: Note that $16 = 2^{5-1}$.

b) Make sure to play the sound in each case.

c) Make up a table showing the number of bits in the quantization scheme, the corresponding distortion ϵ, and your subjective rating of the sound quality.

d) At what point can your ear no longer distinguish the original audio signal from the quantized version?

Consider a periodic function

$$f(x) = f(x + 2A) = x^2, \qquad -A \leq x \leq A.$$

a) Calculate the Fourier series for $f(x)$.

b) In Matlab, use `subplot()` to plot a single figure containing five rows of the first 5 non-zero components of the Fourier series for $f(x)$.

c) In Matlab, use `plot()` to plot a single figure containing three periods of $f(x)$ together with its Fourier series approximation using the first 5 non-zero components.

Homework rules

Submit your results by Friday, October 15 2021 using the web page
`http://zolotarev.fd.cvut.cz/mni`

Solution report should be formally correct (structuring, grammar).

Only `.pdf` files are acceptable. Handwritten solutions and `.doc` and `.docx` files will not be accepted.

Solutions written in TEX (using LyX, Overleaf, whatever) may receive small bonification.