Discrete Fourier Transform

Mathematical Tools for ITS (11MAI)

Mathematical tools, 2021

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Lectue Contents

The derivatives and integrals (as primitive functions) of trigonometric functions are interconnected:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin \ell x = \ell \cos \ell x \Rightarrow \int \cos \ell x \mathrm{d}x = \frac{1}{\ell}\sin \ell x,$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cos \ell x = -\ell \sin \ell x \Rightarrow \int \sin \ell x \mathrm{d}x = -\frac{1}{\ell}\cos \ell x.$$

Product of trigonometric functions

Products of two trigonometric functions are expressible as

$$2\sin \ell x \sin mx = \cos(\ell - m)x - \cos(\ell + m)x,$$
$$2\cos \ell x \cos mx = \cos(\ell - m)x + \cos(\ell + m)x,$$
$$2\sin \ell x \cos mx = \sin(\ell - m)x + \sin(\ell + m)x.$$

Note

If $x \in [0, 2\pi)$ then for $x = \omega_0 t$ we have $t \in [0, T)$.

We have learnt that trigonometric functions $\cos \omega_k t$ and $\sin \omega_k t$ form Fourier basis for T-periodic functions.

Question

Is the basis set of $\cos mx$ and $\sin mx$ for $x \in [0, 2\pi)$ orthogonal?

Assume $\ell, m \in \mathbb{N}$.

We will study the scalar inner products of these functions for $\ell \neq m$ first:

$$
\begin{aligned}
\langle \cos \ell x, \cos mx \rangle &= \int_0^{2\pi} \cos \ell x \cos mx \mathrm{d}x \\
&= \frac{1}{2} \int_0^{2\pi} \cos(\ell - m)x \mathrm{d}x + \frac{1}{2} \int_0^{2\pi} \cos(\ell + m)x \mathrm{d}x \\
&= \frac{1}{2(\ell - m)} \Big[\sin(\ell - m)x\Big]_0^{2\pi} + \frac{1}{2(\ell + m)} \Big[\sin(\ell + m)x\Big]_0^{2\pi} \\
&= \frac{0 - 0}{2(\ell - m)} + \frac{0 - 0}{2(\ell + m)} = 0
\end{aligned}
$$

$$
\begin{aligned}
\langle \sin \ell x, \sin mx \rangle &= \int_0^{2\pi} \sin \ell x \sin mx \mathrm{d}x \\
&= \frac{1}{2}\int_0^{2\pi} \cos(\ell - m)x \mathrm{d}x - \frac{1}{2}\int_0^{2\pi} \cos(\ell + m)x \mathrm{d}x \\
&= \frac{1}{2(\ell - m)}\Big[\sin(\ell - m)x\Big]_0^{2\pi} - \frac{1}{2(\ell + m)}\Big[\sin(\ell + m)x\Big]_0^{2\pi} \\
&= \frac{0-0}{2(\ell - m)} - \frac{0-0}{2(\ell + m)} = 0
\end{aligned}
$$

Orthogonal basis

$$
\begin{aligned}
\langle \sin \ell x, \cos mx \rangle &= \int_0^{2\pi} \sin \ell x \cos mx \mathrm{d}x \\
&= \frac{1}{2} \int_0^{2\pi} \sin(\ell - m)x \mathrm{d}x + \frac{1}{2} \int_0^{2\pi} \sin(\ell + m)x \mathrm{d}x \\
&= -\frac{1}{2(\ell - m)} \Big[\cos(\ell - m)x\Big]_0^{2\pi} - \frac{1}{2(\ell + m)} \Big[\cos(\ell + m)x\Big]_0^{2\pi} \\
&= -\frac{1-1}{2(\ell - m)} - \frac{1-1}{2(\ell + m)} = 0
\end{aligned}
$$

We will study the case $\ell = m$ separately.

$$\langle \sin mx, \cos mx \rangle = \frac{1}{2} \int_0^{2\pi} \sin 2mx \mathrm{d}x = -\frac{1}{4m} \left[\cos 2mx \right]_0^{2\pi}$$
$$= -\frac{1-1}{4m} = 0$$

Normalization

Finally the two cases of basis functions that should result in inner product being 1 if normalised.

$$\langle \cos mx, \cos mx \rangle = \int_0^{2\pi} \cos^2 mx \mathrm{d}x = \int_0^{2\pi} \frac{1 + \cos 2mx}{2} \mathrm{d}x$$
$$= \frac{1}{2}\Big[x\Big]_0^{2\pi} + \frac{1}{2m}\Big[\sin 2mx\Big]_0^{2\pi}$$

$$||\cos mx||^2 = \pi \qquad ||\cos m\omega_0 t||^2 = \frac{T}{2}$$

$$\langle \sin mx, \sin mx \rangle = \int_0^{2\pi} \sin^2 mx \mathrm{d}x = \int_0^{2\pi} \frac{1 - \cos 2mx}{2} \mathrm{d}x$$
$$= \frac{1}{2}\Big[x\Big]_0^{2\pi} - \frac{1}{2m}\Big[\sin 2mx\Big]_0^{2\pi}$$

$$||\sin mx||^2 = \pi \qquad ||\sin m\omega_0 t||^2 = \frac{T}{2}$$

Review: Trigonometric Fourier Series

a) T-periodic signal $x(t)$ representation:

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right)$$

b) basis functions $\cos(k\omega_0 t)$, $\sin(k\omega_0 t)$

c) $a_0 = \dfrac{1}{T} \displaystyle\int_0^T x(t) \mathrm{d}t,$

d) $a_k = \dfrac{\langle x(t), \cos(k\omega_0 t)\rangle}{\langle \cos(k\omega_0 t), \cos(k\omega_0 t)\rangle} \equiv \dfrac{2}{T} \displaystyle\int_0^T x(t) \cos(k\omega_0 t) \mathrm{d}t$

e) $b_k = \dfrac{\langle x(t), \sin(k\omega_0 t)\rangle}{\langle \sin(k\omega_0 t), \sin(k\omega_0 t)\rangle} \equiv \dfrac{2}{T} \displaystyle\int_0^T x(t) \sin(k\omega_0 t) \mathrm{d}t$

Review: Complex Fourier series

a) T-periodic signal representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{j}k\omega_0 t}$$

b) basis functions $\phi_k(t) = \exp(\mathrm{j}k\omega_0 t)$

c) coefficients $c_k = \dfrac{\langle x(t), \phi_k(t)\rangle}{\langle \phi_k(t), \phi_k(t)\rangle} \equiv \dfrac{1}{T}\displaystyle\int_0^T x(t)\mathrm{e}^{-\mathrm{j}k\omega_0 t}\mathrm{d}t$

d) completness of basis functions $\langle \phi_k(t), \phi_\ell(t)\rangle = \dfrac{1}{T}\displaystyle\int_0^T \mathrm{e}^{\mathrm{j}k\omega_0 t}\mathrm{e}^{-\mathrm{j}\ell\omega_0 t}\mathrm{d}t = \delta_{k,\ell}$

Review: Partial sums

a) Fourier series $x(t) = \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{j}\omega_k t}$

b) Partial sum of Fourier Series $x_N(t) = \sum_{k=-M}^{M} c_k \mathrm{e}^{\mathrm{j}\omega_k t}$ for $N = 2M + 1$

From continous to discrete periodic signal

Consider a continuous signal $x(t)$ defined as T-periodical signal, sampled N times during that period at timestamps $t = nT/N$ for $n = 0, 1, 2, \ldots, N-1$. This yields a discretised signal

$$\mathbf{x} = (x_0, x_1, x_2, \ldots, x_{N-1})$$

where $\mathbf{x}$ is a vector in $\mathbb{R}^N$ with N components $x_n = x(nT/N)$.

The sampled signal $\mathbf{x} = (x_0, x_1, x_2, \ldots, x_{N-1})$ can be extended periodically with period N by modular definition

$$x_m = x_{(m \bmod N)}$$

for all $m \in \mathbb{Z}$.

Discrete signal and basis vectors

In order to form the discrete basis vectors we start with exponential basis

$$\phi_k(t) = \mathrm{e}^{\mathrm{j}\omega_k t} = \mathrm{e}^{\mathrm{j}k\omega_0} = \mathrm{e}^{\mathrm{j}2\pi kt/T}$$

and substitute $t \to nT/N$ for $n = 0, 1, 2, \ldots, N-1$.

This yields N components of the basis vector in $\mathbb{C}^N$

$$\phi_{k,n} \equiv \phi_k\left(\frac{nT}{N}\right) = \mathrm{e}^{\mathrm{j}2\pi kn/N}.$$

Discrete signal and basis vectors

The k-th basis vector has the following complex components:

$$\phi_k = \begin{bmatrix} e^{j2\pi k\cdot 0/N} \\ e^{j2\pi k\cdot 1/N} \\ e^{j2\pi k\cdot 2/N} \\ \vdots \\ e^{j2\pi k\cdot (N-1)/N} \end{bmatrix}$$

DFT basis — Orthogonality

We can prove that basis vectors ϕ_k are orthogonal by verifying that $\langle \phi_\ell, \phi_m \rangle = 0$ for all $\ell \neq m$:

$$\langle \phi_\ell, \phi_m \rangle = \sum_{\nu=0}^{N-1} \phi_{\ell,\nu} \overline{\phi_{m,\nu}} = \sum_{\nu=0}^{N-1} \mathrm{e}^{\mathrm{j}2\pi(\ell-m)\nu/N} =$$

$$= \sum_{\nu=0}^{N-1} \left(\mathrm{e}^{\mathrm{j}2\pi(\ell-m)/N} \right)^{\nu}.$$

We have arrived at partial sum of the first N elements for geometric series.

For $\ell \neq m$ we have

$$\langle \phi_\ell, \phi_m \rangle = \frac{1 - \left(\mathrm{e}^{\mathrm{j}2\pi \frac{\ell-m}{N}} \right)^N}{1 - \mathrm{e}^{\mathrm{j}2\pi \frac{\ell-m}{N}}} = \frac{1 - \mathrm{e}^{\mathrm{j}2\pi(\ell-m)}}{1 - \mathrm{e}^{\mathrm{j}2\pi \frac{\ell-m}{N}}} = \frac{1 - 1}{1 - \mathrm{e}^{\mathrm{j}2\pi \frac{\ell-m}{N}}} = 0$$

DFT basis — Scaling factor $||\phi_k||^2$

On $\mathbb{C}^n$ the usual inner product is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\mathsf{T}\overline{\mathbf{y}} = x_1\overline{y}_1 + x_2\overline{y}_2 + \ldots + x_n\overline{y}_n$$

The corresponding norm is

$$||\mathbf{x}||^2 = \langle \mathbf{x}, \mathbf{x} \rangle = x_1\overline{x}_1 + x_2\overline{x}_2 + \cdots + x_n\overline{x}_n = |x_1|^2 + |x_1|^2 + \cdots + |x_n|^2$$

which translates for our basis vector to

$$||\phi_k||^2 = \phi_{k,0}\overline{\phi_{k,0}} + \phi_{k,1}\overline{\phi_{k,1}} + \cdots + \phi_{k,N-1}\overline{\phi_{k,N-1}} = 1 + 1 + \ldots + 1$$

as $\overline{\phi_{k,n}} = \mathrm{e}^{-\mathrm{j}2\pi kn/N}$ is a complex conjugate to $\phi_{k,n} = \mathrm{e}^{\mathrm{j}2\pi kn/N}$ and therefore $\phi_{k,n}\overline{\phi_{k,n}} = 1$, which results in the scaling factor being

$$||\phi_k||^2 = N.$$

Lectue Contents

Strang (2000):

> *The Fourier series is linear algebra in infinite dimensions. The "vectors" are functions $f(t)$; they are projected onto the sines and cosines; that produces the Fourier coefficients a_k and b_k. From this infinite sequence of sines and cosines, multiplied by a_k and b_k, we can reconstruct $f(t)$. That is the classical case, which Fourier dreamt about, but in actual calculations it is the discrete Fourier transform that we compute. Fourier still lives, but in finite dimensions.*

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Definition

Discrete Fourier Transform Let $\mathbf{x} \in \mathbb{C}^N$ be a vector $(x_0, x_1, x_2, \ldots, x_{N-1})$. The discrete Fourier transform (DFT) of $\mathbf{x}$ is the vector $\mathbf{X} \in \mathbb{C}^N$ with components

$$X_k = \langle \mathbf{x}, \phi_k \rangle = \sum_{m=0}^{N-1} x_m \, \mathrm{e}^{-\mathrm{j}2\pi km/N}.$$

Definition

Inverse Discrete Fourier Transform Let $\mathbf{X} \in \mathbb{C}^N$ be a vector $(X_0, X_1, X_2, \ldots, X_{N-1})$. The inverse discrete Fourier transform (IDFT) of $\mathbf{X}$ is the vector $\mathbf{x} \in \mathbb{C}^N$ with components

$$x_k = \frac{\langle \mathbf{X}, \overline{\phi_k} \rangle}{\langle \phi_k, \phi_k \rangle} = \frac{1}{N} \sum_{m=0}^{N-1} X_m \, \mathrm{e}^{\mathrm{j}2\pi km/N}.$$

Definition of the DFT and IDFT

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Discrete Fourier Transform Let $\mathbf{x} \in \mathbb{C}^N$ be a vector $(x_0, x_1, x_2, \ldots, x_{N-1})$. The discrete Fourier transform (DFT) of $\mathbf{x}$ is the vector $\mathbf{X} \in \mathbb{C}^N$ with components

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Definition

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$$x_k = \frac{\langle \mathbf{X}, \overline{\phi_k} \rangle}{\langle \phi_k, \phi_k \rangle} = \frac{1}{N} \sum_{m=0}^{N-1} X_m \, \mathrm{e}^{\mathrm{j}2\pi km/N}.$$

The coefficient X_0/N measures the contribution of the basic waveform $(1, 1, 1, \dots, 1)$ to $\mathbf{x}$. In fact

$$\frac{X_0}{N} = \frac{1}{N}\sum_{m=0}^{N-1} x_m$$

is the average value of $\mathbf{x}$.This coefficient is usually called as the DC coefficient, because it measures the strength of the direct current component of a signal.

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The Fourier Transform can be defined for signals that are

- Discrete or continuous in time
- Finite or infinite duration
- Provided we denote the variable in time domain as $x(t)$, or x_n, the transformed variables in frequency domain are correspondingly $X(j\omega)$ or X_k.

This unification results in four cases.

An overview of Fourier transforms

	continuous in time	discrete in time periodic in frequency
continuous in frequency	$x(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$ $X(j\omega) = \int\limits_{-\infty}^{\infty} e^{-j\omega t}x(t)dt$ Fourier transform	$x(n) = \frac{T}{2\pi}\int\limits_{-\pi/T}^{+\pi/T} X(e^{j\omega T})e^{jk\omega T}d\omega$ $X(e^{j\omega T}) = \sum\limits_{n=-\infty}^{\infty} x(n)e^{-jk\omega T}$ Fourier transform $t = nT$ (DTFT)
discrete in frequency periodic in time	$x(t) = \sum\limits_{k=-\infty}^{\infty} X(k)e^{jk\omega_0 t}$ $X(k) = \frac{\omega_0}{2\pi}\int\limits_{-\pi/\omega_0}^{\pi/\omega_0} x(t)e^{-jn\omega_0 t}dt$ Fourier series	$x(n) = \frac{1}{N}\sum\limits_{k=0}^{N-1} X(k)e^{(j2\pi/N)kn}$ $X(k) = \sum\limits_{n=0}^{N-1} x(n)e^{-(j2\pi/N)kn}$ Discrete Fourier transform (DFT)

The DFT consists of inner products of the input sequence $(x_n)_{n=0}^{N-1}$ stored as $\mathbf{x} = (x_0, x_1, x_2, \ldots, x_{N-1})$ with N basis vectors representing sampled complex sinusoidal sections

$$\phi_k = (\phi_{k,n})_{n=0}^{N-1} = \left(\mathrm{e}^{\mathrm{j}2\pi kn/N}\right)_{n=0}^{N-1}$$

yielding for $k = 0, 1, 2, \ldots, N-1$

$$X_k = \langle \mathbf{x}, \phi_k \rangle = \mathbf{x}^\mathsf{T} \overline{\phi_k} = \sum_{m=0}^{N-1} x_m \, \mathrm{e}^{-\mathrm{j}2\pi km/N}.$$

An overview of discrete Fourier Transform

By collecting the DFT output samples into a column vector, we have

$$\underbrace{\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_{N-1} \end{bmatrix}}_{\mathbf{X}} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \overline{w_N^1} & \overline{w_N^2} & \cdots & \overline{w_N^{N-1}} \\ 1 & \overline{w_N^2} & \overline{w_N^4} & \cdots & \overline{w_N^{2(N-1)}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \overline{w_N^{N-1}} & \overline{w_N^{2(N-1)}} & \cdots & \overline{w_N^{(N-1)(N-1)}} \end{bmatrix}}_{\mathbf{\Phi}_N^*} \underbrace{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix}}_{\mathbf{x}}$$

Finally we can write matrix representation as

$$\mathbf{X} = \mathbf{\Phi}_N^* \mathbf{x}.$$

An overview of discrete Fourier Transform

The matrix $\mathbf{W}_N^* = \overline{(\mathbf{W}_N)}^{\mathsf{T}}$ denotes the Hermitian transpose of the complex matrix $\mathbf{W}_N$. It can be shown that

$$\mathbf{W}_N^* \times \mathbf{W}_N = \begin{bmatrix} N & 0 & 0 & \cdots & 0 \\ 0 & N & 0 & \cdots & 0 \\ 0 & 0 & N & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & N \end{bmatrix} = N \cdot 1$$

and consequently the inversion of the Eq. (26) is

$$\mathbf{x} = \frac{1}{N}\mathbf{W}_N\mathbf{X}.$$

Some practical comments

If the number of digital samples in each time slice is a power of 2, one can use a faster version of the DFT known as the fast Fourier transform (FFT)

The FFT assumes that the samples being analyzed comprise one cycle of a periodic wave. In most cases it is not the case and analysis will contain many spurious frequencies not actually present in the signal.

Sample fast enough and long enough!

To recognize details in frequency domain use spectral interpolation.

What is aliasing?

It is easiest to describe in terms of a visual sampling:

We all know and love movies. If you have ever watched a western and seen the wheel of a rolling wagon appear to be going backwards, you have witnessed aliasing. The movie's frame rate is not adequate to describe the rotational frequency of the wheel, and our eyes are deceived by the misinformation.

The Nyquist Theorem tells us that we can successfully sample and play back frequency components up to one-half the sampling frequency.

Aliasing is the term used to describe what happens when we try to record and play back frequencies higher than one-half the sampling rate.

What is aliasing?

Consider a digital audio system with a sample rate of 48 KHz, recording a steadily rising sine wave tone. At lower frequency, the tone is sampled with many points per cycle. As the tone rises in frequency, the cycles get shorter and fewer and fewer points are available to describe it. At a frequency of 24 KHz, only two sample points are available per cycle, and we are at the limit of what Nyquist says we can do.

Still, those two frequency points are adequate, in a theoretical world, to recreate the tone after conversion back to analog and low-pass filtering.

What is aliasing?

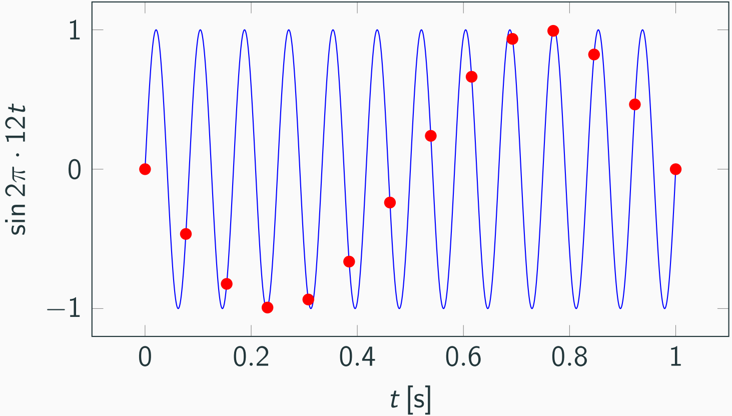
But, if the tone continues to rise, the number of samples per cycle is not adequate to describe the waveform, and the inadequate description is equivalent to one describing a lower frequency tone – this is aliasing.

In fact, the tone seems to reflect around the 24 KHz point:

- A 25 KHz tone becomes indistinguishable from a 23 KHz tone.
- A 30 KHz tone becomes an 18 KHz tone.

Aliasing due to insufficient sampling

The following figure illustrates what happens if a signal is sampled at regular time intervals that are slightly below the period of the original signal.



Zero padding

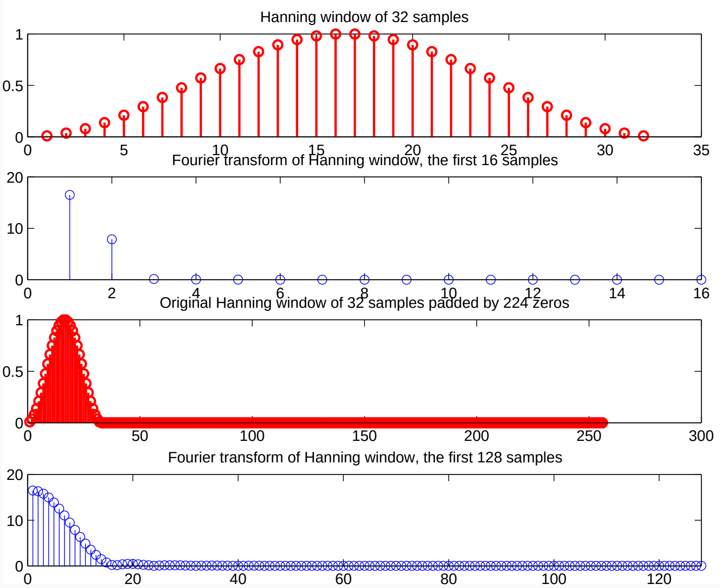
Zero padding consists of appending zeros to a signal. It maps a length N signal to a length $M > N$ signal. M does not need to be an integer multiple of N.

Zero padding in the time domain gives spectral interpolation in the frequency domain.

Similarly, zero padding in the frequency domain gives bandlimited interpolation in the time domain. This is how ideal sampling rate conversion is accomplished.

Usually we use FFT which requires signals of length $M = 2^m$ which means we chose the number of zeros equal to $2^m - N$.

Zero padding: How does it work?



Lectue Contents

Trigonometric formulae

Vector spaces of continuous and discrete waveforms

Discrete Fourier Transform – DFT

Revision of sampled signals

Definition (Nyquist-Shannon Sampling Theorem, 1927)

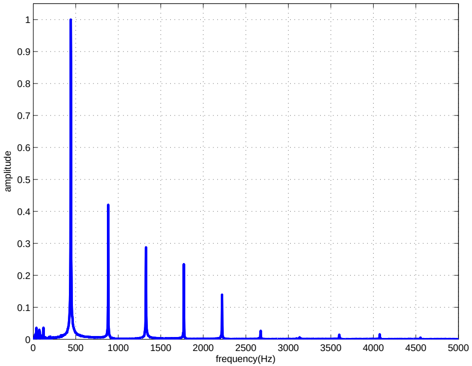
It is possible precisely to reconstruct a continuous-time signal from its samples, given that

a) the signal is bandlimited;

b) the sampling frequency f_s is greater than twice the signal bandwidth.

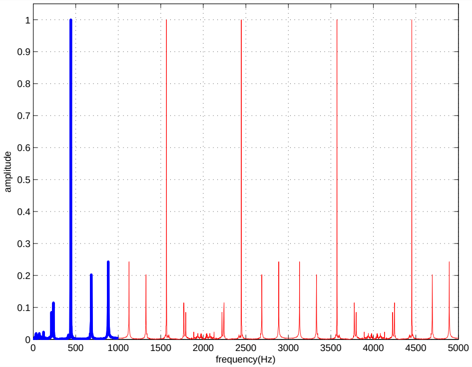
Aliasing in Audio

- The initial sound is a numerically synthesized piano-tone at 440 Hz. The sampling frequency is of 44.1 kHz (CD-quality).
- The harmonic frequencies at multiple of the fundamental tone (440 Hz) are clearly visible.



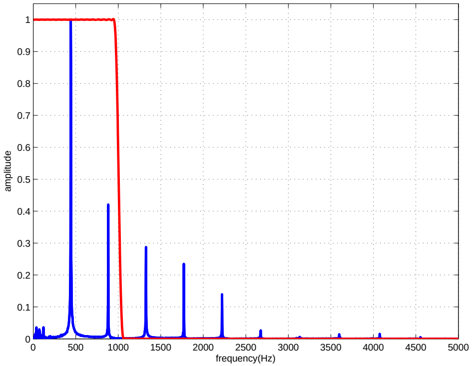
Aliasing in Audio

- The sound will be resampled at 2 kHz, without precautions against aliasing. The tone sounds rather strange.
- The aliasing is visible on the graphs as a “warping” of the frequencies against a “mirror” at the Nyquist frequency 1 kHz.



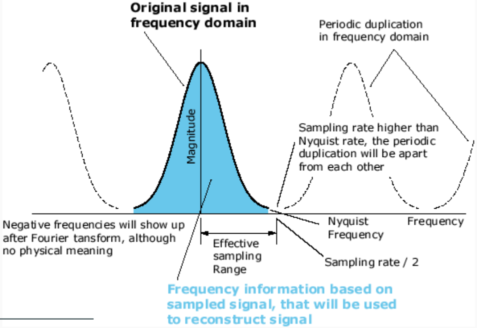
Aliasing in Audio

- In order to avoid aliasing, the spectrum of the signal should be zero at frequencies higher than the Nyquist frequency before resampling. A low-pass filter is used to achieve this



...for a digital signal processing with DFT there are limits:

- The signal must be band-limited. This means there is a frequency above which the signal is zero.
- Hence the maximum useable frequency in the DFT is $f_s/2$ - the Nyquist[1] frequency!



[1] Harry Nyquist 1889-1976