From Fourier Series to Analysis of Non-stationary Signals – I

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Theorem (Fundamental Theorem of Algebra)
Every nth-order polynomial possesses exactly n complex roots

This is a very powerful algebraic tool. It says that given any polynomial

$$\begin{aligned} P_n(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \\ &\equiv \sum_{i=0}^{n} a_i x^i, \end{aligned}$$

Fundamental Theorem of Algebra

we can always rewrite it as

$$P_n(x) = a_n(x - x_n)(x - x_{n-1}) \cdots (x - x_2)(x - x_1)$$
$$\equiv a_n \prod_{i=1}^{n} (x - x_i)$$

where the points x_i are the polynomial roots and they may be real or complex.

Example (Roots)
Consider the second-order polynomial

$$P_2(x) = x^2 + 7x + 12.$$

The polynomial is second-order because the highest power of x is 2 and is also monic because its leading coefficient of x^2, is $a_2 = 1$.

By the fundamental theorem of algebra there are exactly two roots x_1 and x_2, and we can write

$$P_2(x) = (x - x_1)(x - x_2).$$

Show that the roots are $x_1 = -3$ and $x_2 = -4$.

The factored form of this simple example is

$$P_2(x) = x^2 + 7x + 12 = (x - x_1)(x - x_2) = (x + 3)(x + 4).$$

Note that polynomial factorization rewrites a monic nth-order polynomial as the product of n first-order monic polynomials, each of which contributes one root (zero) to the product.

This factoring process is often used when working in digital signal processing (DSP).

Factoring Polynomials in Matlab

Factoring can be also performed by MATLAB commands

```
p2=[1 7 12];
roots(p2)
```

Example 1: Find the factors of following polynomials:

- $P_3(x) = x^3 + 2x^2 + 2x + 1$
- $P_2(x) = 9\,x^2 + a^2$
- $P_4(x) = x^4 - 1$

Factoring Polynomials

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In order to study the roots of $P_4(x) = x^4 - 1$ using MATLAB, you can write a command creating the polynomial

```
p4=[1 0 0 0 -1],
```

follwed by commands

```
roots(p4),
```

and

```
zplane(p4).
```

which gives you a plot of the roots in the complex domain.

Taylor Series

Taylor's Theorem with Remainder

A Taylor series is a series expansion of a function about a point.

It is a local approximation.

A one-dimensional Taylor series is an expansion of a real function $f(x)$, which is $(n+1)$-times differentiable, about a point $x = a$ is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + R_n(x) \tag{1}$$

where

$$R_n(x) = \frac{1}{(n+1)!} \int_a^x f^{(n+1)}(a)(x-a)^{n+1}.$$

The last term $R_n(x)$ is called the remainder, or error term.

A Taylor polynomial of order n is a partial sum of a Taylor series
no reminder!

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3. \quad (2)$$

If $a = 0$, the expansion is also known as a Maclaurin series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3. \quad (3)$$

Example 2: Evaluate the first five terms of Taylor series of $f(x) = \dfrac{1}{1-x}$

$$
\begin{aligned}
f(x) &= \frac{1}{1-x} & f(0) &= 1 \\
f'(x) &= \frac{1}{(1-x)^2} & f'(0) &= 1 \\
f''(x) &= \frac{2}{(1-x)^3} & f''(0) &= 2 \\
f'''(x) &= \frac{2 \times 3}{(1-x)^4} & f'''(0) &= 6 \\
f''''(x) &= \frac{6 \times 4}{(1-x)^5} & f''''(0) &= 24
\end{aligned}
$$

And as

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4$$

we have

$$\frac{1}{1-x} \approx 1 + x + x^2 + x^3 + x^4.$$

Do you remember the formula for geometric series ?!

Two symbols to be defined

. . . e (Euler's number) and $\sqrt{-1}$.

$$\mathrm{i} \equiv \sqrt{-1}$$

$$\mathrm{e} \equiv \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.71828182845905\ldots$$

The first, $\mathrm{i} = \sqrt{-1}$, is the basis for complex numbers, called **imaginary unit**.

The second, $\mathrm{e} = 2.718\ldots$, is a transcendental real number defined by the above limit. It is the base of the natural logarithm.

Example
Approximations of $f(x)$ up to 3 terms

- $f(x) = e^x$
- $f(x) = \sin x$
- $f(x) = \cos x$

Taylor Polynomials

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{4}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

If we introduce imaginary unit $\imath x$ in Eq. (4) we obtain

$$e^{\imath x} = 1 + \frac{\imath x}{1!} + \frac{(\imath x)^2}{2!} + \frac{(\imath x)^3}{3!} + \frac{(\imath x)^4}{4!} + \dots \tag{5}$$

Role of imaginary exponent

For imaginary unit we have

$$\imath^1 = \sqrt{-1}$$
$$\imath^2 = -1$$
$$\imath^3 = -\imath$$
$$\imath^4 = 1$$

and Equation (5) has form

$$e^{\imath x} = 1 + \imath\frac{x}{1!} - \frac{x^2}{2!} - \imath\frac{x^3}{3!} + \frac{x^4}{4!} + \dots \tag{6}$$

Euler identity

It can be easily identified with

$$e^{ix} = 1 + i\frac{x}{1!} - \frac{x^2}{2!} - \imath\frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \equiv \cos x + i\sin x \qquad (7)$$

The result is the famous Euler's formula (1743 Opera Omnia, vol. 14, p. 142)

$$e^{\imath x} = \cos x + \imath \sin x$$

Euler's identity

Euler's identity is the key to understanding the meaning of expressions like

$$f(\omega_k T) \equiv e^{i\omega_k T} = \cos(\omega_k T) + \imath \sin(\omega_k T).$$

We will see later that such an expression defines a sampled complex sinusoid.

MATLAB project

MATLAB project

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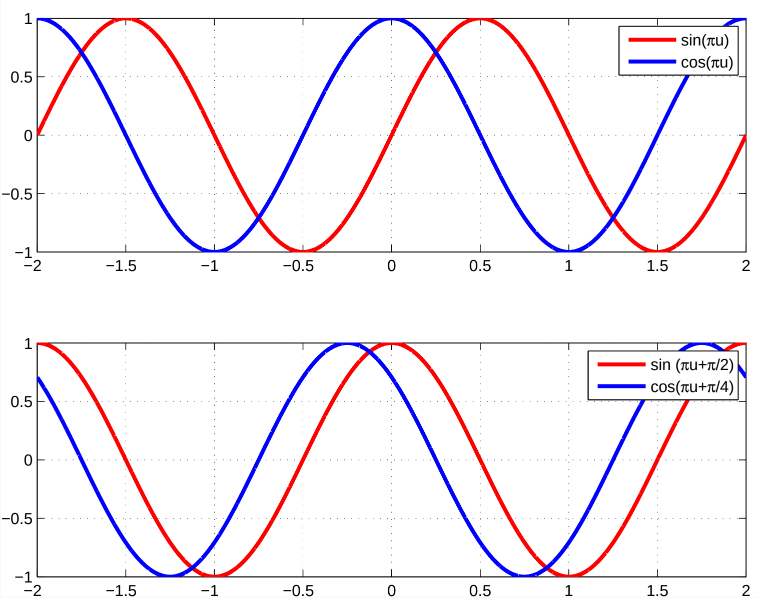
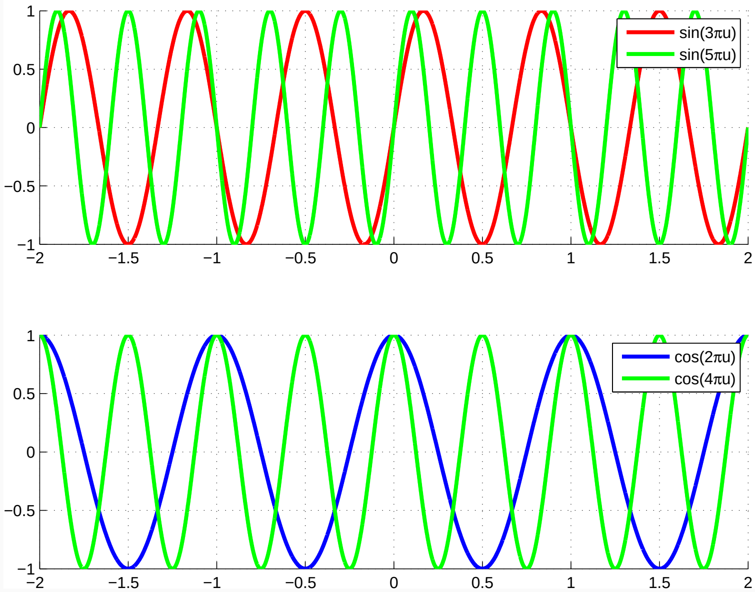
1. Using MATLAB plot the graphs of the sine and cosine functions, $\sin(\pi u)$ and $\cos(\pi u)$ within the interval $-2 \leq u \leq 2$.
2. Plot graphs of the functions $\sin(\pi u + \pi/2)$ and $\cos(\pi u + \pi/4)$ within the interval $-2 \leq u \leq 2$.
3. Plot graphs of the functions $\sin(3\pi u)$ and $\sin(5\pi u)$ within the interval $-2 \leq u \leq 2$.
4. Display axes, add legends to all graphs.
5. Save every output as a PNG, EPS, and Windows EMF file.


sin(πu)
cos(πu)
1
0.5
0
−0.5
−1
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
sin (πu+π/2)
cos(πu+π/4)
1
0.5
0
−0.5
−1
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2


sin(3πu)
sin(5πu)
cos(2πu)
cos(4πu)
1
0.5
0
−0.5
−1
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2

Sample solution – Sine and cosine function


```
u=linspace(-2,2,4000); % 4000 points from −2 to 2
ys0=sin(pi*u);
yc0=cos(pi*u); % sine and cosine
ys=sin(pi*u+pi/2);
yc=cos(pi*u+pi/4); % ... with phase shift

figure(1); % not strictly necessary

subplot(2,1,1); % 2 rows, 1 column, 1st row
plot(u, ys0, 'LineWidth', 2.5, 'Color', 'r');
hold on
plot(u, yc0, 'LineWidth', 2.5, 'Color', 'b');
legend('sin($\pi$␣u)','cos($\pi$␣u)');
grid on
hold off
```

Sample solution – Sine and cosine function

```
subplot(2,1,2); % 2 rows, 1 column, 2nd row
% alternative color definition as RGB triplet
plot(u, ys, 'LineWidth', 2.5, 'Color', [1 0 0])
hold on
plot(u, yc, 'LineWidth', 2.5, 'Color', [0 0 1])
legend('sin($\pi$␣u␣+␣$\pi$/4)','cos($\pi$␣u␣+␣$\pi$/4)')
grid on
hold off
```

Homework 1

Derive the formulae for factoring the following polynomials:

$$P_{2n}(x) = x^{2n} \pm 1$$
$$P_{2n+1}(x) = x^{2n+1} \pm 1$$

1. Check your results using MATLAB command `roots` for finding the roots of a polynomial.
2. Plot the roots of polynomials of degree $2n = 16$ and $2n = 32$ using MATLAB command `zplane`.
3. Follow the symmetrical properties of the roots. Report on what do you observe.
4. Deliver your results by Wednesday, October 9 2019 to the web page `http://zolotarev.fd.cvut.cz/mni`.

Homework rules

Solution report should be formally correct (structuring, grammar).

Only `.pdf` files are acceptable. Handwritten solutions and `.doc` and `.docx` files will not be accepted.

Solutions written in $\TeX$ (using LyX, Overleaf, whatever) may receive small bonification.