From Fourier Series to Analysis of Non-stationary Signals – III

Mathematical tools, 2019

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Review vector spaces

Recall vectors in n-dimensional space $\mathbb{R}^n$. Each such vector $\mathbf{u}$ can be uniquely represented as a linear combination of n unit basis vectors $\mathbf{e_1}, \ldots, \mathbf{e_n}$:

$$\mathbf{u} = \alpha_1 \mathbf{e_1} + \alpha_2 \mathbf{e_2} + \ldots + \alpha_n \mathbf{e_n},$$

Q: How to compute the values of $\alpha_i \in \mathbb{R}$?

Definition (Inner product)

Operation that assigns a non-negative scalar a to a pair of vectors $\mathbf{u}$ and $\mathbf{v}$, denoted $(\mathbf{u}, \mathbf{v})$, where:

1. $(\mathbf{u} + \mathbf{w}, \mathbf{v}) = (\mathbf{u}, \mathbf{v}) + (\mathbf{w}, \mathbf{v})$
2. $(\alpha\mathbf{u}, \mathbf{v}) = \alpha(\mathbf{u}, \mathbf{v})$
3. $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$
4. $(\mathbf{v}, \mathbf{v}) \geq 0$, $(\mathbf{v}, \mathbf{v}) = 0 \Leftrightarrow \mathbf{v} \equiv \mathbf{0}$

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Definition (Inner product space)

Inner product space is a vector space with inner product operation defined.

For inner product space we still have

$$\mathbf{u} = \alpha_1 \mathbf{e_1} + \alpha_2 \mathbf{e_2} + \ldots + \alpha_n \mathbf{e_n},$$

and in addition $\alpha_i \in \mathbb{R}$ can be computed using the inner product $(\cdot, \cdot)$ as

$$\alpha_i = (\mathbf{u}, \mathbf{e_i})$$

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Definition (Orthornormal vectors)

Vectors $\mathbf{e}_i$ are orthonormal if they are

- normalized – $\forall i : \mathbf{e}_i . \mathbf{e}_i = \|\mathbf{e}_i\|^2 = 1$
- orthogonal – $\forall i \neq j : \mathbf{e}_i . \mathbf{e}_j = (\mathbf{e}_i, \mathbf{e}_j) = 0$

Example

Draw addition of two vectors in two dimensional space $\mathbb{R}^2$:

$$\mathbf{u} = 3\mathbf{e_1} + 4\mathbf{e_2}$$

$$\mathbf{v} = -2\mathbf{e_1} + 3\mathbf{e_2}$$

and make them normalized.

Vectors are objects that can be added together and multiplied by scalars - vector space:

- if $\mathbf{u} = \sum_{i=1}^{n} \alpha_i \mathbf{e}_i$ and $\mathbf{v} = \sum_{i=1}^{n} \beta_i \mathbf{e}_i \Rightarrow$

$$\mathbf{u} + \mathbf{v} = \sum_{i=1}^{n} (\alpha_i + \beta_i)\mathbf{e}_i$$

- if $\mathbf{u} = \sum_{i=1}^{n} \alpha_i \mathbf{e}_i$ and λ is scalar $\Rightarrow$

$$\lambda \mathbf{u} = \sum_{i=1}^{n} \lambda \alpha_i \mathbf{e}_i$$

Vector space of continuous-time signals

We have already studied the space of continuous-time signals. We can easily verify:

- we can form the sum of any two signals $x_1(t)$ and $x_2(t)$ to obtain another signal

$$x(t) = x_1(t) + x_2(t)$$

- we can multiply any signal $x(t)$ by a constant λ to obtain another signal

$$y(t) = \lambda x(t)$$

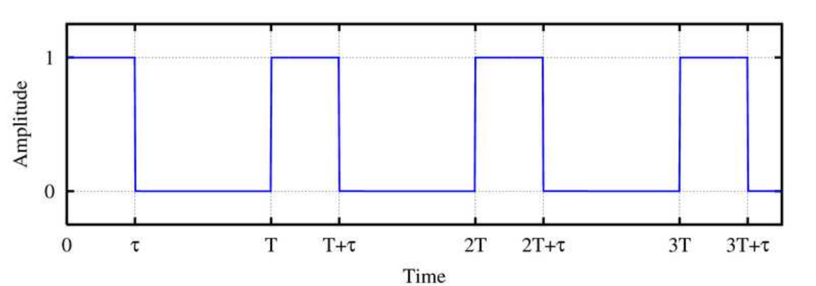
Unlike the n-dimensional space $\mathbb{R}^n$, the vector space of all continuous-time signals is infinite-dimensional.

Vector space of periodic signals

Consider now periodic signals; any such signal $x(t)$ satisfies periodicity condition:

$$x(t + T) = x(t) \text{ for all } t$$

for given period T.



Vector space of periodic signals

It is easy to see that periodic signals form a vector space:

- if $x_1(t)$ and $x_2(t)$ are periodic, then

  $$x(t+T) = x_1(t+T) + x_2(t+T) = x_1(t) + x_2(t) = x(t)$$

  is also periodic with the same period T
- if $x_1(t)$ is periodic and λ is scalar, then

  $$y(t+T) = \lambda x(t+T) = \lambda x(t) = y(t)$$

  is a scaled version of $x(t)$ being also periodic with period T

Vector space of periodic signals

If we impose even more conditions on periodic signals – the Dirichlet conditions, which hold for all signals encountered in practice, then we can represent signals as infinite linear combinations of orthogonal and normalized vectors.

- A function satisfying Dirichlet conditions must have right and left limits at each point of discontinuity:

$$x(t+) = \lim_{\tau \to t+} x(\tau) \text{ and } x(t-) = \lim_{\tau \to t-} x(\tau)$$

- The Dirichlet theorem says in particular that the Fourier series for $x(t)$ converges and is equal to $x(t) = \frac{x(t+)+x(t-)}{2}$ wherever $x(t)$ is continuous.

Complete orthonormal systems of functions

Definition (Inner product of T-periodic signals)

We can define the inner product of two T-periodic signals $x_1(t)$ and $x_2(t)$ as

$$(x_1(t), x_2(t)) = \int_0^T x_1(t)x_2(t)\,\mathrm{d}t$$

We can integrate over any complete period, i.e. from $-\frac{T}{2}$ to $-\frac{T}{2}$

$$(x_1(t), x_2(t)) = \int_{-\frac{T}{2}}^{\frac{T}{2}} x_1(t)x_2(t)\,\mathrm{d}t.$$

Complete orthonormal systems

Then we can take any sequence of T-periodic functions $\{\phi_j(t)\}_{j\in\mathbb{N}}$ that are

- normalized – $(\phi_j(t), \phi_j(t)) = \|\phi_j(t)\|^2 = \int_0^T \phi_j^2(t)\mathrm{d}t$
- orthogonal – $(\phi_j(t), \phi_k(t)) = \int_0^T \phi_j(t)\phi_k(t)\mathrm{d}t = 0$ for $j \neq k$
- complete – if a signal $x(t)$ is such that

$$(\phi_j(t), x(t)) = \int_0^T \phi_j(t)x(t)\mathrm{d}t = 0$$

for all j, then $x(t) = 0$

Trigonometric and complex exponential Fourier Series

Let $\{\phi_j(t)\}_{j\in\mathbb{N}}$ be a complete, orthonormal set of functions. Then any well-behaved T-periodic signal $x(t)$ can be uniquely represented as an infinite series

$$x(t) = \sum_{j=0}^{\infty} \alpha_j \phi_j(t)$$

This is called the Fourier series representation of $x(t)$. The scalars (numbers) α_j are called the Fourier coefficients of $x(t)$ with respect to $\{\phi_j(t)\}_{j\in\mathbb{N}}$ and are computed as follows:

$$\alpha_j = (\phi_j(t), x(t)) = \int_0^T \phi_j(t)x(t)dt.$$

In analogy to vectors in n-dimensional space, you can think of α_j as the projection of $x(t)$ in the direction of $\phi_j(t)$.

Proof:

To derive the formula for α_j, write

$$x(t)\phi_k(t) = \sum_{j=0}^{\infty} \alpha_j \phi_j(t)\phi_k(t)$$

and then integrate over a period

$$(\phi_k(t), x(t)) = \int_0^T \phi_k(t)x(t)\,\mathrm{d}t = \int_0^T \sum_{j=0}^{\infty} \alpha_j \phi_j(t)\phi_k(t)\,\mathrm{d}t.$$

For convergent series we can integrate term by term and

$$\int_0^T \sum_{j=0}^{\infty} \alpha_j \phi_j(t) \phi_k(t)\,\mathrm{d}t = \sum_{j=0}^{\infty} \alpha_j \int_0^T \phi_j(t)\phi_k(t)\mathrm{d}t = \sum_{j=0}^{\infty} \alpha_j \delta_{j,k} = \alpha_k$$

Here and in following evaluation we will use Kronecker delta which is defined as $\delta_{j,k} = 0$ for $j \neq k$ and $\delta_{k,k} = 1$ and which indicates that $\{\phi_j(t)\}_{j=0}^{\infty}$ form an orthonormal system of functions.

It can be also proved that, as the functions $\{\phi_j(t)\}_{j=0}^{\infty}$ form a complete orthonormal system, the partial sums of the Fourier series

$$x(t) = \sum_{j=0}^{\infty} \alpha_j \phi_j(t)$$

converge to $x(t)$ in the following sense (L_2-convergence):

$$\lim_{N\to\infty} \int_0^T \left(x(t) - \sum_{j=0}^{N} \alpha_j \phi_j(t) \right)^2 \mathrm{d}t = 0.$$

Similarly to the case of Taylor polynomial, we can use (with some care for discontinuities) the partial sum

$$x(t) \approx \sum_{j=0}^{N} \alpha_j \phi_j(t)$$

to approximate $x(t)$.

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The sequence of T-periodic functions $\{\phi_k(t)\}_{k=0}^{\infty}$ defined for $m = 1, 2, \ldots$ by

1. $\phi_0(t) = \dfrac{1}{\sqrt{T}}$
2. $\phi_{2m-1}(t) = \sqrt{\dfrac{2}{T}} \cos(m \omega_0 t)$
3. $\phi_{2m}(t) = \sqrt{\dfrac{2}{T}} \sin(m \omega_0 t)$

is complete and orthonormal. Here $\omega_0 = \dfrac{2\pi}{T}$ is called fundamental frequency.

Note the first few functional elements of the sequence from the previous slide (without scaling factors):

$$\{1, \cos t, \sin t, \cos 2t, \sin 2t, \cos 3t, \sin 3t, \ldots\}$$

Trigonometric Fourier Series

Common way of writing down the trigonometric Fourier series of $x(t)$ is following:

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$$

The Fourier coefficients can be computed as follows:

1. $a_0 = \frac{1}{T} \int_0^T x(t)\,\mathrm{d}t$
2. $a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\,\mathrm{d}t$
3. $b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\,\mathrm{d}t$

To relate this to the orthonormal representation in terms of the $\{\phi_j(t)\}_{j\in\mathbb{N}}$, we note that we can write

1. $$a_0 = \frac{1}{T}\int_0^T x(t)\,\mathrm{d}t = \frac{1}{\sqrt{T}}\int_0^T x(t)\,\frac{1}{\sqrt{T}}\,\mathrm{d}t = \frac{1}{\sqrt{T}}\int_0^T x(t)\phi_0(t)\,\mathrm{d}t = \frac{1}{\sqrt{T}}\,\alpha_0$$
2. $a_k = \cdots$
3. $b_k = \cdots$
4. $$x(t) = a_0 + \sum_{k=1}^{\infty} a_k\cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k\sin(k\omega_0 t) \equiv \sum_{j=0}^{\infty}\alpha_j\phi_j(t).$$

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To relate this to the orthonormal representation in terms of the $\{\phi_j(t)\}_{j\in\mathbb{N}}$, we note that we can write

1. $a_0 = \frac{1}{\sqrt{T}}\alpha_0$
2. $a_k = \cdots$
3. $b_k = \cdots$
4. $x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t) \equiv \sum_{j=0}^{\infty} \alpha_j \phi_j(t).$

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1. $a_0 = \frac{1}{\sqrt{T}}\,\alpha_0$
2. $a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\,\mathrm{d}t = \sqrt{\frac{2}{T}}\int_0^T x(t)\sqrt{\frac{2}{T}}\cos(k\,\omega_0 t)\,\mathrm{d}t$
   $= \sqrt{\frac{2}{T}}\int_0^T x(t)\phi_{2k-1}(t)\,\mathrm{d}t = \sqrt{\frac{2}{T}}\,\alpha_{2k-1}$
3. $b_k = \cdots$
4. $x(t) = a_0 + \sum_{k=1}^{\infty} a_k\cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k\sin(k\omega_0 t) \equiv \sum_{j=0}^{\infty}\alpha_j\phi_j(t).$

Trigonometric Fourier Series

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3. $b_k = \cdots$
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2. $a_k = \sqrt{\frac{2}{T}}\alpha_{2k-1}$
3. $b_k = \frac{2}{T}\int_0^T x(t)\sqrt{\frac{2}{T}}\,\mathrm{d}t = \sqrt{\frac{2}{T}}\int_0^T x(t)\sqrt{\frac{2}{T}}\sqrt{\frac{2}{T}}\,\mathrm{d}t$
   $= \sqrt{\frac{2}{T}}\int_0^T x(t)\phi_{2k}(t)\,\mathrm{d}t = \sqrt{\frac{2}{T}}\alpha_{2k}$
4. $x(t) = a_0 + \sum_{k=1}^{\infty} a_k\cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k\sin(k\omega_0 t) \equiv \sum_{j=0}^{\infty}\alpha_j\phi_j(t).$

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4. $x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t) \equiv \sum_{j=0}^{\infty} \alpha_j \phi_j(t).$

In symmetrical cases:

1. if $x(t)$ is an even function, i.e., $x(t) = x(-t)$ for all t, then all its sine Fourier coefficients are zero:

$$b_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\sin(k\omega_0 t)\,\mathrm{d}t = 0$$

2. if $x(t)$ is an odd function, i.e., $x(t) = -x(-t)$, then all its cosine Fourier coefficients are zero:

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\cos(k\omega_0 t)\,\mathrm{d}t = 0$$

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Theorem (Fourier series of an even function)
Fourier series of an even function $f(t) = f(-t)$ consists of the constant and cosine terms

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t),$$

where $\omega_0 = \dfrac{2\pi}{T}$.

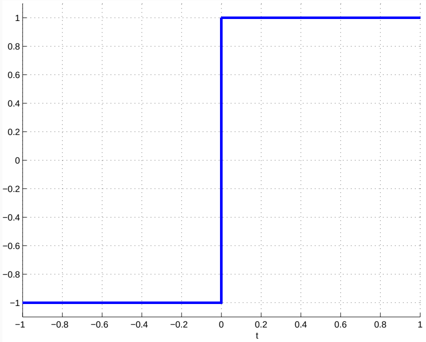
Theorem (Fourier series of an even function)

Fourier series of an odd function $f(t) = -f(-t)$ consists of the sine terms

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t),$$

where $\omega_0 = \dfrac{2\pi}{T}$.

Example 1: Consider a periodic signal $x(t) = x(t+T)$ given by repeating the square wave



Note, that here $T = 2$!

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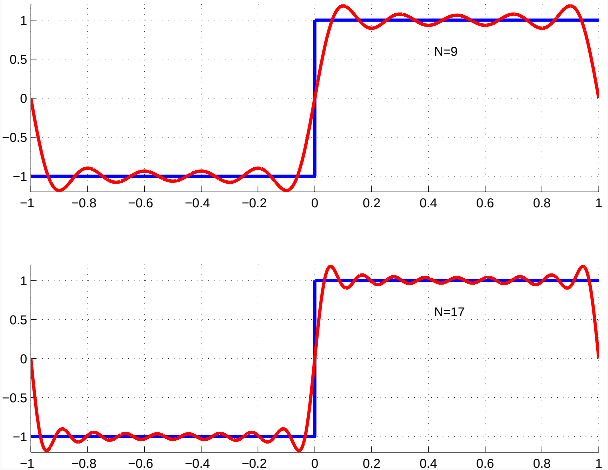
Solution:

1. the signal has odd symmetry $\Rightarrow$ all $a_k = 0$
2. $b_k = \frac{2}{T}\int_{-1}^{1} x(t)\sin(k\omega_0 t)\,\mathrm{d}t$
$$= \frac{2}{T}\int_{-1}^{0}(-1)\sin(k\omega_0 t)\,\mathrm{d}t + \frac{2}{T}\int_{0}^{1}(+1)\sin(k\omega_0 t)\,\mathrm{d}t$$
$$= \frac{1}{k\pi}\Big[cos(k\pi t)\Big]_{-1}^{0} - \frac{1}{k\pi}\Big[cos(k\pi t)\Big]_{0}^{1}$$
$$= \frac{2}{k\pi}(1-\cos(k\pi)) = \frac{4}{k\pi}\sin^2(\frac{k\pi}{2})$$
3. For $k = 2m-1$ is $b_k = \frac{4}{k\pi}\sin^2(\frac{k\pi}{2}) = \frac{4}{(2m-1)\pi}$
4. $x(t) = \sum_{m=1}^{\infty}\frac{4}{(2m-1)\pi}\sin((2m-1)\,\pi t)$

Partial sums

$$x_N(t) = \sum_{m=1}^{N} \frac{4}{(2m-1)\pi} \sin(2m-1)\,\pi t$$



The Fourier series (over/under)shoots the actual value of $x(t)$ at points of discontinuity regardless of degree N.

Complex exponentials

Another useful complete orthonormal set is accomplished by the complex exponentials:

1. $\phi_k(t) = \dfrac{1}{\sqrt{T}} \exp(j\,k\omega_0 t)$ for $k = \ldots - 2, -1, 0, 1, 2, \ldots$
2. these functions are complex-valued, and we have to evaluate the inner product as

$$(x_1(t), x_2(t)) = \int_0^T x_1(t) x_2^*(t)\,\mathrm{d}t,$$

where $x_2^*(t)$ denotes complex conjugation

Complex exponential Fourier series

1. $(\phi_k(t), \phi_\ell(t)) = \dfrac{1}{T} \displaystyle\int_0^T \exp(j\, k\omega_0 t) \exp(-j\, \ell\omega_0 t)\, \mathrm{d}t = \delta_{k,\ell}$
2. $x(t) = \displaystyle\sum_{k=-\infty}^{\infty} c_k \exp(j\, k\omega_0 t)$
3. $c_k = \dfrac{1}{T} \displaystyle\int_0^T x(t) \exp(-j\, k\omega_0 t))\, \mathrm{d}t$
4. as in trigonometric case $\omega_0 = \dfrac{2\pi}{T}$

Project

Project 1 – Half-wave rectified sinusoid

Find the Fourier series representation for the half-wave rectified sinusoid.

$$f(t) = \begin{cases} \sin\left(\frac{2\pi t}{T}\right) & \text{if} \quad 0 \leq t \leq \frac{T}{2} \\ 0 & \text{if} \quad \frac{T}{2} \leq t \leq T \end{cases}$$

- Calculate the coefficients a_k and b_k using identities

$$2\sin \ell x \sin mx = \cos(\ell - m)x - \cos(\ell + m)x,$$
$$2\sin \ell x \cos mx = \sin(\ell - m)x + \sin(\ell + m)x.$$

- Plot the first 5 components of the Fourier series using Matlab.

Find the Fourier series representation for the sawtooth

$$f(t) = f(t + T) = t$$

if $-T/2 \leq t \leq T/2$.

- As the function $f(t)$ is odd, the coefficients $a_k = 0$. Calculate coefficients b_k.
- Plot the first 5 components of the Fourier series using Matlab.

Homework

- Calculate the Fourier series for

$$f(x) = x^2$$

  if $-A \leq x \leq A$.
- Compare the results for space periodicity $f(x + 2A) = f(x)$ with those obtained for time periodicity $f(t + T) = f(t)$.
- In Matlab, use `subplot()` to plot five rows of the first 5 components of the Fourier series.