From Fourier Series to Analysis of Non-stationary Signals – IV

Mathematical tools, 2019

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Trigonometric formulae

Integrals of trigonometric functions

The derivatives and integrals (as primitive functions) of trigonometric functions are interconnected:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin \ell x = \ell \cos \ell x \Rightarrow \int \cos \ell x \,\mathrm{d}x = \frac{1}{\ell}\sin \ell x,$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos \ell x = -\ell \sin \ell x \Rightarrow \int \sin \ell x \,\mathrm{d}x = -\frac{1}{\ell}\cos \ell x.$$

Product of trigonometric functions

Products of two trigonometric functions are expressible as

$$2\sin \ell x \sin mx = \cos(\ell - m)x - \cos(\ell + m)x,$$
$$2\cos \ell x \cos mx = \cos(\ell - m)x + \cos(\ell + m)x,$$
$$2\sin \ell x \cos mx = \sin(\ell - m)x + \sin(\ell + m)x$$

Note

If $x \in (0, 2\pi)$ then for $x = \omega_0 t$ we have $t \in (0, T)$.

We have learnt that trigonometric functions $\cos m\omega_0 t$ and $\sin m\omega_0 t$ form Fourier basis for T-periodic functions.

Question

Is the basis $\cos mx$ and $\sin mx$ orthogonal?

Orthogonal basis

We will study the scalar inner products of these functions for $\ell \neq m$ first:

$$
\begin{aligned}
(\cos \ell x, \cos mx) &= \int_0^{2\pi} \cos \ell x \cos mx \, \mathrm{d}x \\
&= \frac{1}{2} \int_0^{2\pi} \cos(\ell - m)x \, \mathrm{d}x + \frac{1}{2} \int_0^{2\pi} \cos(\ell + m)x \, \mathrm{d}x \\
&= \frac{1}{2(\ell - m)} \Big[\sin(\ell - m)x \Big]_0^{2\pi} + \frac{1}{2(\ell + m)} \Big[\sin(\ell + m)x \Big]_0^{2\pi} \\
&= \frac{0 - 0}{2(\ell - m)} + \frac{0 - 0}{2(\ell + m)} = 0
\end{aligned}
$$

Orthogonal basis

$$
\begin{aligned}
(\sin \ell x, \sin mx) &= \int_0^{2\pi} \sin \ell x \sin mx \, \mathrm{d}x \\
&= \frac{1}{2} \int_0^{2\pi} \cos(\ell - m)x \, \mathrm{d}x - \frac{1}{2} \int_0^{2\pi} \cos(\ell + m)x \, \mathrm{d}x \\
&= \frac{1}{2(\ell - m)} \Big[\sin(\ell - m)x \Big]_0^{2\pi} - \frac{1}{2(\ell + m)} \Big[\sin(\ell + m)x \Big]_0^{2\pi} \\
&= \frac{0 - 0}{2(\ell - m)} - \frac{0 - 0}{2(\ell + m)} = 0
\end{aligned}
$$

Orthogonal basis

$$
\begin{aligned}
(\sin \ell x, \cos mx) &= \int_0^{2\pi} \sin \ell x \cos mx \, \mathrm{d}x \\
&= \frac{1}{2}\int_0^{2\pi} \sin(\ell - m)x \, \mathrm{d}x + \frac{1}{2}\int_0^{2\pi} \sin(\ell + m)x \, \mathrm{d}x \\
&= -\frac{1}{2(\ell - m)}\Big[\cos(\ell - m)x\Big]_0^{2\pi} - \frac{1}{2(\ell + m)}\Big[\cos(\ell + m)x\Big] \\
&= -\frac{1-1}{2(\ell - m)} - \frac{1-1}{2(\ell + m)} = 0 \\
(\sin mx, \cos mx) &= \frac{1}{2}\int_0^{2\pi} \sin 2mx \, \mathrm{d}x \\
&= -\frac{1}{4m}\Big[\cos 2mx\Big]_0^{2\pi} = 0 \quad \text{for } \ell = m
\end{aligned}
$$

Normalization

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We will study the case $\ell = m$ separately

$$(\cos mx, \cos mx) = \int_0^{2\pi} \cos^2 mx \, \mathrm{d}x = \int_0^{2\pi} \frac{1 + \cos 2mx}{2} \mathrm{d}x$$
$$= \frac{1}{2} [x]_0^{2\pi} + \frac{1}{2m} [\sin 2mx]_0^{2\pi}$$

$$||\cos mx||^2 = \pi \qquad ||\cos m\omega_0 t||^2 = \frac{T}{2}$$

$$(\sin mx, \sin mx) = \int_0^{2\pi} \sin^2 mx \, \mathrm{d}x = \int_0^{2\pi} \frac{1 - \cos 2mx}{2} \mathrm{d}x$$
$$= \frac{1}{2} [x]_0^{2\pi} - \frac{1}{2m} [\sin 2mx]_0^{2\pi}$$

$$||\sin mx||^2 = \pi \qquad ||\sin m\omega_0 t||^2 = \frac{T}{2}$$

Vector space of continuous basic waveforms

Trigonometric Fourier Series

1. T-periodic signal $x(t)$ representation:

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$$

2. basis vectors $\cos(k\omega_0 t)$, $\sin(k\omega_0 t)$
3. $a_0 = \dfrac{1}{T}\displaystyle\int_0^T x(t)\,\mathrm{d}t,$

   $a_k = \dfrac{(x(t), \cos(k\omega_0 t))}{(\cos(k\omega_0 t), \cos(k\omega_0 t))} \equiv \dfrac{2}{T}\displaystyle\int_0^T x(t)\cos(k\omega_0 t)\,\mathrm{d}t$
4. $b_k = \dfrac{(x(t), \sin(k\omega_0 t))}{(\sin(k\omega_0 t), \sin(k\omega_0 t))} \equiv \dfrac{2}{T}\displaystyle\int_0^T x(t)\sin(k\omega_0 t)\,\mathrm{d}t$

Continuous signal and basis vectors

1. T-periodic signal representation $x(t) = \sum_{k=-\infty}^{\infty} c_k \exp(j\,k\omega_0 t)$
2. basis vector $\phi_k(t) = \exp(j\,k\omega_0 t)$
3. scalar product
$$c_k = \frac{(x(t), \phi_k(t))}{(\phi_k(t), \phi_k(t))} \equiv \frac{1}{T}\int_0^T x(t)\exp(-j\,k\omega_0 t))dt$$
4. completness of basis vectors
$$(\phi_k(t), \phi_\ell(t)) = \frac{1}{T}\int_0^T \exp(j\,k\omega_0 t)\exp(-j\,\ell\omega_0 t)dt = \delta_{k,\ell}$$

Continuous signal and basis vectors

1. Fourier series $x(t) = \sum_{k=-\infty}^{\infty} c_k \exp(j\, k\omega_0 t)$
2. Partial sum of Fourier Series $x_N(t) = \sum_{k=-M}^{M} c_k \exp(j\, k\omega_0 t)$ for $N = 2M + 1$

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Definition (Dirichlet kernel)

Dirichlet kernels are the partial sums of exponential functions

$$D_M(\omega_0 t) = \sum_{k=-M}^{M} \exp(j\, k\omega_0 t) = 1 + 2\sum_{k=1}^{M} \cos(k\omega_0 t).$$

Show that $D_M(\omega_0 t) = \dfrac{\sin((M+1/2)\omega_0 t)}{\sin(\omega_0 t/2)}$.

Theorem (Convolution of Dirichlet kernel)

The convolution of $D_M(t)$ with an arbitrary T-periodic function $f(t) = f(t+T)$ is the M-th degree Fourier series approximation to $f(t)$.

$$D_M(t) * f(t) \equiv \frac{1}{T} \int_{-T/2}^{T/2} D_M(t-\tau) f(\tau) d\tau = \sum_{k=-M}^{M} c_k \exp(j\, k\omega_0 t),$$

where $c_k = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \exp(-j\, k\omega_0 t)\, dt.$

Vector space of discrete basic waveforms

Discrete signal and basis vectors

Consider a continuous signal $x(t)$ defined as T-periodical signal, sampled at the N times $t = nT/N$ for $n = 0, 1, 2, \ldots, N-1$. This yields discretised signal

$$\mathbf{x} = (x_0, x_1, x_2 \ldots, x_{N-1})$$

where $\mathbf{x}$ is a vector in $\mathbb{R}^N$ with N components $x_n = x(nT/N)$. The sampled signal $\mathbf{x} = (x_0, x_1, x_2 \ldots, x_{N-1})$ can be extended periodically with period N by modular definition

$$x_m = x_{m \bmod N}$$

for all m outside the range $0 \leq m \leq N-1$.

Discrete signal and basis vectors

In order to form the discrete basis vectors we substitute in

$$\phi_k(t) = \exp(jk\omega_0 t) = \exp\left(j\frac{2\pi kt}{T}\right)$$

the discrete time $t = nT/N$ yielding N components of the basis vector

$$\phi_{k,n} \equiv \phi_k\left(\frac{nT}{N}\right) = \exp\left(j\frac{2\pi kn}{N}\right).$$

Basis vector has complex components

$$\phi_k = \begin{bmatrix} \exp(j\frac{2\pi k0}{N}) \\ \exp(j\frac{2\pi k1}{N}) \\ \exp(j\frac{2\pi k2}{N}) \\ \vdots \\ \exp(j\frac{2\pi k(N-1)}{N}) \end{bmatrix}$$

Discrete signal and basis vectors

On $\mathbb{C}^n$ the usual scalar (inner) product is

$$(\mathbf{x}, \mathbf{y}) = x_1\overline{y}_1 + x_2\overline{y}_2 + \ldots + x_n\overline{y}_n$$

The corresponding norm is

$$||\mathbf{x}||^2 = (\mathbf{x}, \mathbf{x}) = x_1\overline{x}_1 + x_2\overline{x}_2 + \ldots + x_n\overline{x}_n = |x_1|^2 + |x_1|^2 + \ldots + |x_n|^2$$

which translates for our basis vector to

$$\begin{aligned} ||\boldsymbol{\phi}_k||^2 &= \phi_{k,0}\overline{\phi_{k,0}} + \phi_{k,1}\overline{\phi_{k,1}} + \ldots + \phi_{k,N-1}\overline{\phi_{k,N-1}} \\ &= 1 + 1 + \ldots + 1 = N \end{aligned}$$

as $\overline{\phi_{k,n}} = \exp(-j\frac{2\pi kn}{N})$ is a complex conjugate to $\phi_{k,n} = \exp(j\frac{2\pi kn}{N})$.

Discrete signal and basis vectors

We can prove that basis vectors are orthogonal using scalar product (ϕ_k, ϕ_ℓ) is zero for $k \neq \ell$. Actually

$$(\phi_k, \phi_\ell) = \sum_{\nu=0}^{N-1} \phi_{k,\nu}\overline{\phi_{\ell,\nu}} = \sum_{\nu=0}^{N-1} \exp(j\frac{2\pi\,(k-\ell)\,\nu}{N}) = \\ = \sum_{\nu=0}^{N-1} \left(\exp(j\frac{2\pi\,(k-\ell)}{N})\right)^{\nu}.$$

We have arrived to geometric series. Its partial sum for $k \neq \ell$ is

$$(\phi_k, \phi_\ell) = \frac{1 - \left(\exp(j\frac{2\pi\,(k-\ell)}{N})\right)^{N}}{1 - \exp(j\frac{2\pi\,(k-\ell)}{N})} = \frac{1 - \exp(j2\pi\,(k-\ell))}{1 - \exp(j\frac{2\pi\,(k-\ell)}{N})} = 0$$

Discrete Fourier Transform – DFT

Definition of the DFT

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1. Let $\mathbf{x} \in C^N$ be a vector $(x_0, x_1, x_2, \ldots, x_{N-1})$. The discrete Fourier transform (DFT) of $\mathbf{x}$ is the vector $\mathbf{X} \in \mathbb{C}^N$ with components

$$X_k = (\mathbf{x}, \Phi_k) = \sum_{m=0}^{N-1} x_m \exp(-j\frac{2\pi\, k\, m}{N}).$$

2. Let $\mathbf{X} \in C^N$ be a vector $(X_0, X_1, X_2, \ldots, X_{N-1})$. The inverse discrete Fourier transform (IDFT) of $\mathbf{X}$ is the vector $\mathbf{x} \in \mathbb{C}^N$ with components

$$x_k = \frac{(\mathbf{X}, \Phi_{-k})}{(\Phi_k, \Phi_k)} = \frac{1}{N} \sum_{m=0}^{N-1} X_m \exp(j\frac{2\pi\, k\, m}{N}).$$

The coefficient X_0/N measures the contribution of the basic waveform $(1, 1, 1, \dots, 1)$ to $\mathbf{x}$. In fact

$$\frac{X_0}{N} = \frac{1}{N}\sum_{m=0}^{N-1} x_m$$

is the average value of $\mathbf{x}$.This coefficient is usually called as the dc coefficient, because it measures the strength of the direct current component of a signal.

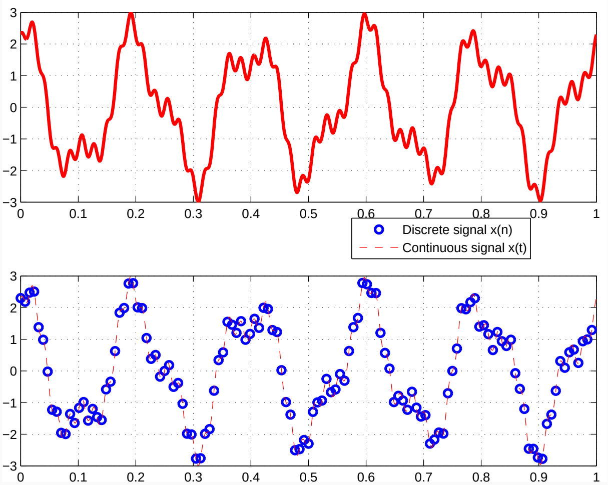
Project

Example
Consider the analog signal

$$x(t) = 2.0\cos(2\pi\, 5t) + 0.8\sin(2\pi\, 12t) + 0.3\cos(2\pi\, 47t)$$

on the interval $t \in (0, 1)$. Sample this signal with period $\tau = 1/128\,\mathrm{s}$ and obtain sample vector $\mathbf{x} = (x_0, x_1, x_2, \ldots, x_{127})$.

1. Make MATLAB m-file which plots signals $x(t)$ and $\mathbf{x}$
2. Using definition of the DFT find $\mathbf{X}$.
3. Use MATLAB command fft(x) to compute DFT of $\mathbf{X}$.
4. Make MATLAB m-file which computes DFT of $\mathbf{x}$ and plots signal and its spectrum.
5. Compute IDFT of the $\mathbf{X}$ and compare it with the original signal $x(t)$.


3
2
1
0
−1
−2
−3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Discrete signal x(n)
Continuous signal x(t)

Solution of example 1

```
clear
% plots original and sampled signal
t = linspace(0,1,1001);
x = 2.0*cos(2*pi*5*t) + 0.8*sin(2*pi*12*t) + ...
    0.3*cos(2*pi*47*t);
N = 128; % number of samples
tdelta = 1/N; % sampling period
ts(1) = 0;
xs(1) = x(1);
for k = 2:1:N
    ts(k) = (k-1)*tdelta;
    xs(k) = 2.0*cos(2*pi*5*(k-1)*tdelta) + ...
            0.8*sin(2*pi*12*(k-1)*tdelta) + ...
            0.3*cos(2*pi*47*(k-1)*tdelta);
end
```

Solution of example 1

```
figure(1);
subplot(2,1,1);
plot(t,x,'LineWidth',2.5,'Color',[1 0 0]);
grid on;
subplot(2,1,2);
plot(ts, xs,'o','LineWidth',2.0,'Color',[0 0 1]);
hold on;
plot(t,x,'--','Color',[1 0 0]);
grid on;
legend('Discrete␣signal␣x(n)','Continuous␣signal␣x(t)');
hold off;
pause
```

Homework

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- Start MATLAB. Load in the "ding" audio signal with command `y=wavread('ding.wav');` The audio signal is stereo one and can be decoupled into two channels by `y1=y(:,1); y2=y(:,2);`. The sampling rate is 22 050 Herz, and the signal contains 20 191 samples. If we consider this signal as sampled on an interval $(0, T)$, then $T = 20191/22050 \approx 0.9157$ seconds.
- Compute the DFT of the signal with `Y1=fft(y1);` and `Y2=fft(y2);`. Display the magnitude of the Fourier transform with `plot(abs(Y1))` or `plot(abs(Y2))`. The DFT is of length 20 191 and symmetric about center.

MATLAB project with audio signal

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- Since MATLAB indexes from 1, the DFT coefficient Y_k is actually `Y(k+1)` in MATLAB ! Also Y_k corresponds to frequency $k/T = k/0.9157$ and so `Y(k+1)` corresponds to $f_k = (k-1)/T = (k-1)/0.9157$.
- You can plot only the first half of the DFT with `plot(abs(Y1(1:6441)))` or `plot(abs(Y2(1:6441)))`. Use the data cursor button on plot window to pick out the frequency and amplitude of the two (obviously) largest components in the spectrum. Compute the actual value of each significant frequency in Herz.

- Let f_1, f_2 denote these frequencies in Herz, and let A_1, A_2 denote the corresponding amplitudes from the plot. Define these variables in MATLAB.
- Generate a new signal using only these frequencies, sampled at 22 050 Herz on the interval $(0, 1)$ with

```
t = [0:1/22050:1];
y12 = (A1*sin(2*pi*f1*t) + A2*sin(2*pi*f2*t))/(A1+A2)
```

- Play the original sound with `sound(y1)` and the synthesized version `sound(y12)`. Repeat the experiment with sound of the second channel `sound(y2)`. Note that our synthesis does not take into account the phase information at these frequencies.
- Does the artificial generated signal reproduce `ding.wav` correctly? Compare the quality!