From Fourier Series to Analysis of Non-stationary Signals – IX

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December 2, 2019

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Nonstationary Signals and Analysis

There are several approaches available for addressing non-stationary signals:

- The Short Time Fourier Transform has in principle excellent frequency resolution, with the time resolution varying with the length of the window, which for some non-stationary signals fails to provide information about substantial time events.

- The Wavelet Transform possesses excellent time resolution, but in principle it has poorer frequency resolution than the Short Time Fourier Transform. The continuous wavelet functions are not generally orthogonal, so the signal reconstruction is ill-defined, while the Discrete Wavelet Transform (DWT) forms orthogonal or biorthogonal bases which do not produce redundant components in signal analysis.

- The Hilbert-Huang Transform, in spite of its good performance in analysing non-linear, non-stationary signals, has no inversion (e.g. there is no process for reconstructing the original signal).

Introduction to Wavelets

1. Wavelet bases like Fourier bases reveal the signal properties through the coefficients of its expansion

2. Wavelets are well localized and few coefficients are needed to represent local transient structure.

3. Wavelet basis defines a sparse representation of piecewise regular signal which may include transients and singularities.

4. In images, the wavelet coefficients are located in the neighborhood of edges and irregular textures. Wavelets form bases for the image compression standard JPEG 2000.

5. Wavelets hardly provides frequency content of the signal.

The story began in 1910, when Alfred Haar constructed a piecewise constant function

$$\psi(t) = \begin{cases} 1 & \text{if } 0 \leq t < 0.5 \\ -1 & \text{if } 0.5 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

the dilatations and translations of which generate an orthonormal basis

$$\left\{ \psi_{\ell,n}(t) = \sqrt{2^\ell}\psi\left(2^\ell\, t - n\right) \right\}_{\ell,n \in \mathbb{Z}^2} \tag{1}$$
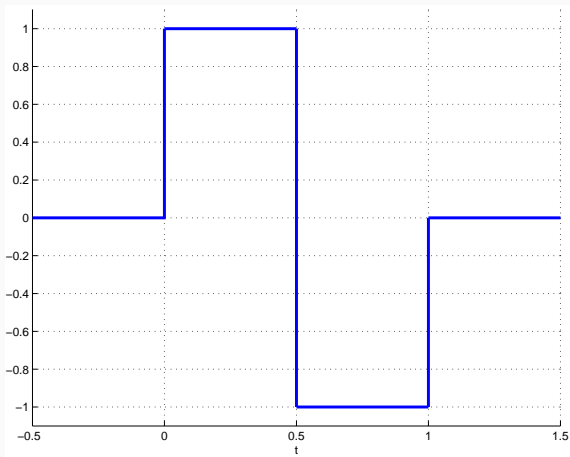
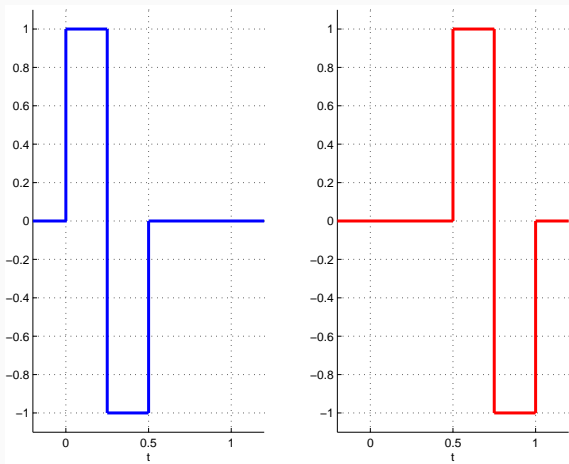**Figure 1:** Mother Haar wavelet $\psi_{0,0}(t)$

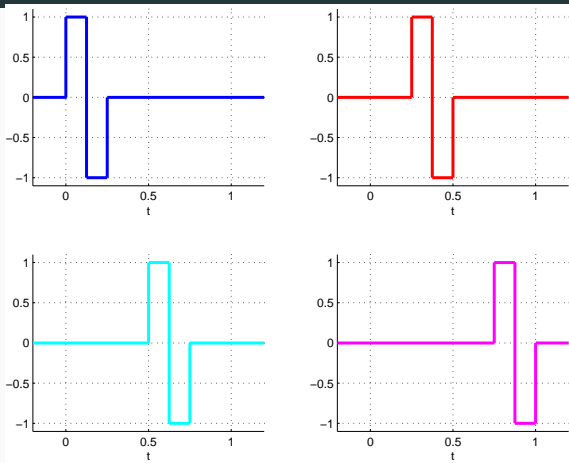**Figure 2:** Scaled Haar wavelets $\psi_{1,0}(t)$ and $\psi_{1,1}(t)$

Figure 3: Scaled Haar wavelets $\psi_{2,0}(t)$, $\psi_{2,1}(t)$, $\psi_{2,2}(t)$, $\psi_{2,3}(t)$

A general form of the Haar wavelets can be written explicitly as

$$\psi_{\ell,n}(t) = \begin{cases} +\sqrt{2^\ell} & \text{if } \dfrac{n}{2^\ell} \leq t < \dfrac{n+0.5}{2^\ell}, \\ -\sqrt{2^\ell} & \text{if } \dfrac{n+0.5}{2^\ell} \leq t < \dfrac{n+1}{2^\ell}, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Each Haar wavelet $\psi_{\ell,n}(t)$ has a zero average over its support $\langle \dfrac{n}{2^\ell}, \dfrac{n+1}{2^\ell} \rangle$ of length $2^{-\ell}$. For any fixed ℓ the function $\psi_{\ell,n}(t)$ is simply $\psi_{\ell,0}(t)$ translated $\dfrac{n}{2^{\ell+1}}$ units to the right.

Definition (Function support)

The support of a function $f(t)$ defined on $\mathbb{R}$ is the closure of the set on which $f(t)$ is nonzero. A function $f(t)$ is said to be supported in a set $\mathcal{A} \subseteq \mathbb{R}$, if the support of $f(t)$ is contained in $\mathcal{A}$.

If a function $f(t)$ is supported on a bounded interval $[a, b]$, then $f(t)$ is said to have compact support on $[a, b]$.

Scalar product

$$\langle f(t), g(t) \rangle = \int_0^1 f(t)\, g^*(t)\, \mathrm{d}t$$

Norm

$$||f(t)||^2 = \langle f(t), f(t) \rangle = \int_0^1 f(t)\, f^*(t)\, \mathrm{d}t$$

Orthogonality

$$\langle \psi_{\ell,n}(t), \psi_{m,n}(t) \rangle = \int_0^1 \psi_{\ell,n}(t)\, \psi_{m,n}(t)\, \mathrm{d}t = 2^\ell \delta_{\ell,m}$$

If we define the Haar scaling function as

$$\phi(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1, \\ 0 & \text{otherwise,} \end{cases}$$

then any function $f(t)$ can be approximated in arbitrary precision L on $(0, 1)$ as

$$f_{L+1}(t) = c_0 \phi(t) + \sum_{\ell=0}^{L} \sum_{n=0}^{2^L - 1} c_{\ell,n} \psi_{\ell,n}(t),$$

where

$$c_0 = \langle f(t), \phi(t) \rangle = \int_0^1 f(t) \, \mathrm{d}t$$

$$c_{\ell,n} = \langle f(t), \psi_{\ell,n}(t) \rangle = \sqrt{2^\ell} \int_0^1 f(t) \psi_{\ell,n}(t) \, \mathrm{d}t$$

Proposition (Average value of $f(t)$)

For any $f(t) \in (0, 1)$ with finite energy $\int_0^1 f(t) f^*(t) \, dt$, the constant value of the approximation $f_{L+1}(t)$ on any interval $\langle n \, 2^{-L}, (n+1) \, 2^{-L} \rangle$ is the average value of $f(t)$ on that interval.

Proof by induction!

Let $f(t) = t\,(1-t)\,(2-t)$ on interval $0 \le t \le 1$. Using Haar wavelets, find approximation of this function for $L = 0$ and $L = 1$.

As the scale function $\phi(t)$ is equal to 1 for the whole interval, we have for $L = 0$

$$c_0 = \int_0^1 f(t)\,\mathrm{d}t = \int_0^1 (t^3 - 3\,t^2 + 2\,t)\,\mathrm{d}t = \left[\frac{t^4}{4} - t^3 + t^2\right]_0^1 = \frac{1}{4}$$

and the only wavelet scaling coefficient

$$c_{0,0} = \sqrt{2^0} \int_0^1 f(t)\psi_{0,0}(t)\,\mathrm{d}t =$$

$$= \int_0^{1/2} f(t)\cdot(+1)\,\mathrm{d}t + \int_{1/2}^1 f(t)\cdot(-1)\,\mathrm{d}t = a_1 + a_2$$

$$a_1 = \int_0^{1/2} f(t) \cdot (+1)\, \mathrm{d}t = \int_0^{1/2} (t^3 - 3\,t^2 + 2\,t)\, \mathrm{d}t =$$

$$= \left[\frac{t^4}{4} - t^3 + t^2 \right]_0^{1/2} = \frac{1}{4}$$

$$a_2 = \int_0^{1/2} f(t) \cdot (-1)\, \mathrm{d}t = -\int_{1/2}^1 (t^3 - 3\,t^2 + 2\,t)\, \mathrm{d}t =$$

$$= -\left[\frac{t^4}{4} - t^3 + t^2 \right]_{1/2}^1 = \frac{1}{4}$$

$$a2 = \int_0^1 dt\, f(t)\psi_{0,0}(t) = -2 \int_{1/2}^1 dt(t^3 - 3\,t^2 + 2\,t)$$

$$= -2 \left[\frac{t^4}{4} - t^3 + t^2\right]_{1/2}^1 = -2 \left(\frac{1}{4} - \frac{9}{64}\right) = -\frac{7}{32} \tag{3}$$

so that

$$f_0(t) = \begin{cases} \dfrac{9}{32} & \text{if } 0 \le t < 0.5 \\ (-1) \times -\dfrac{7}{32} = \dfrac{7}{32} & \text{if } 0.5 \le t < 1.0 \end{cases} \tag{4}$$
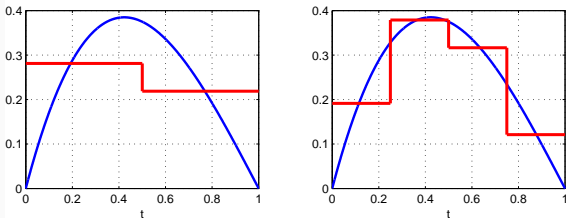
Figure 4: Haar expansion using all wavelets $\psi_{0,0}(t)$, $\psi_{1,0}(t)$ and $\psi_{1,1}(t)$

Derive by integration the constant values of

$$f_1(t) = \left[\frac{49}{256} \ \frac{95}{256} \ \frac{81}{256} \ \frac{31}{256} \right]$$

and plot the diagrams.

- There is enormous number of wavelets having names after researchers in 80s and 90s.

- To name few of them Yves Mayer, Stephane Mallat, Ingrid Daubechies . . .

- For example, a simple request on Web of Knowledge for "Wavelet Transform" revealed close to 750 publications in 2010/2011.

Typing `waveinfo('wname')` you get information about wavelets

Wavelet Family Short Name	Wavelet Family Name
'haar'	Haar wavelet
'db'	Daubechies wavelets
'sym'	Symlets
'bior'	Biorthogonal wavelets
'rbior'	Reverse biorthogonal wavelets
'meyr'	Meyer wavelet
'dmey'	Discrete approximation of Meyer wavelet
'gaus'	Gaussian wavelets
'mexh'	Mexican hat wavelet
'morl'	Morlet wavelet

A note to your compositions

We have already talked about sound improvements.

You can furhter improve the audio by including some partials to an original tone, i.e. to the tone A_4 (440 Hz) we will add also 440 Hz, 880 Hz, ..., $440k$ [Hz].

If you make a `spectrogram` of a real piano tone A you can see relative duration of harmonics
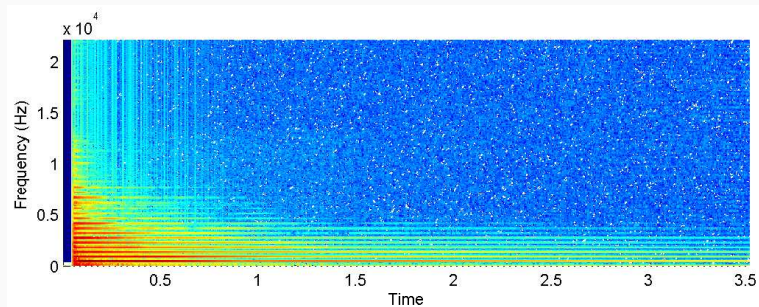
MATLAB project

1. Start MATLAB and create an artificial piecewise constant signal with

```matlab
y = zeros(256,1);
y(100:160) = 1.0;
```

2. The Discrete Wavelet Transform is provided by MATLAB under a variety of extension modes. We can perform a one-stage wavelet decomposition of the signal for example with

```matlab
[cA, cD] = dwt(y,'bior2.2');
```

3. Here `cA` is a vector of the approximation coefficients and `cD` the detailed coefficients.

4. The `bior2.2` argument specifies the wavelet to be used, which in this case corresponds to La Gall 5/3 wavelet.

5. You can execute `waveinfo('bior')` to see other orthogonal wavelets.

6. Plot the vectors `cA` and `cD`.

   Since the signal is almost constant, `cA` looks like an approximately half-length version of the original signal. There are some distortions at the jumps.

   The vector `cD` is mostly zero except at the discontinuity which contains high frequency energy.

7. Since the signal is implicitly extended at the boundary by symmetric reflection, there are no discontinuities there.

8. We can reconstruct an approximation to the original full length using only coefficients `cA` by

   ```
   y2 = upcoef('a',cA,'bior2.2',1);
   ```

9. Plot `y2`!

10. Change the extension mode to periodic extension with command `dwtmode('per')`

11. Periodic extension introduces discontinuities at the signal boundaries. This is obvious in the plots `cA` and `cD` and in reconstruction `y2`

12. Repeat the above steps with another wavelet.

```
% Set effective support and grid parameters.
lb = -8; ub = 8; n = 1024;
% Meyer wavelet and scaling functions.
[phi,psi,x] = meyer(lb,ub,n);
subplot(211)
plot(x,psi), grid
title('Meyer wavelet')
subplot(212)
plot(x,phi), grid
title('Meyer scaling function')
```
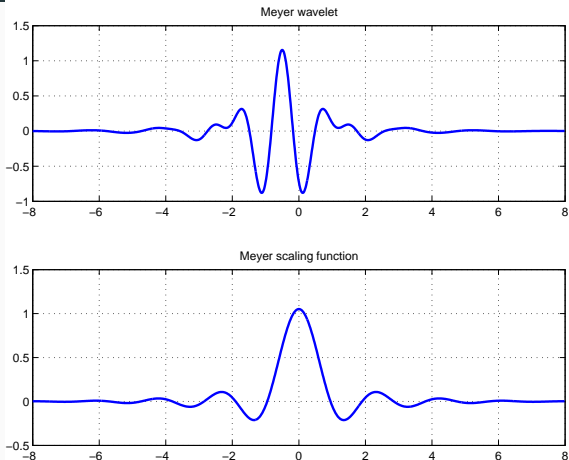
Figure 5: Meyer wavelets