Transfer function.
Stability of continuous-time systems.

Modeling Systems and Processes (11MSP)

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Solution of second order differential equation

Differential equation

$$\frac{d^2}{dt^2} y(t) + 2a \frac{d}{dt} y(t) + (a^2 + b^2)y(t) = u(t)$$

with initial conditions

$$y(0) = c_1 \quad a \quad y'(0) = c_2,$$

we solve by Laplace transform.
Solution of second order differential equation

Because

\[ \mathcal{L} \left\{ \frac{d}{dt} y(t) \right\} = pY(p) - y(0), \]
\[ \mathcal{L} \left\{ \frac{d^2}{dt^2} y(t) \right\} = p^2 Y(p) - py(0) - \frac{d}{dt} y(0), \]

we find the Laplace transform of the differential equation. The algebraic equation is then

\[ p^2 Y(p) - py(0) - y'(0) + 2a(pY(p) - y(0)) + (a^2 + b^2)Y(p) = U(p). \]
Solution of second order differential equation

Previous equation is solved with respect to the output variable $Y(p)$ and we get

$$(p^2 + 2ap + (a^2 + b^2)) Y(p) = U(p) + py(0) + y'(0) + 2ay(0)$$

or

$$Y(p) = \frac{U(p) + c_2 + (p + 2a)c_1}{(p + a + ib)(p + a - ib)}$$
Transfer function

The transfer function $H(p)$ is defined as the ratio of the output $Y(p)$ to the input $U(p)$

$$H(p) = \frac{Y(p)}{U(p)}$$

for zero initial conditions and, therefore,

$$H(p) = \frac{Y(p)}{U(p)} = \frac{1}{(p + a + ib)(p + a - ib)}$$
The impulse response is determined as the inverse Laplace transform of the Transfer function

\[
\mathcal{L}^{-1}\{H(p)\} = \mathcal{L}^{-1}\left\{\frac{1}{(p + a)^2 + b^2}\right\} = \frac{1}{b}e^{-at}\sin bt
\]
Step response $s(t)$ is determined as the inverse Laplace transform of $s(t) = \mathcal{L}^{-1}\left\{H(p) \frac{1}{p}\right\}$

\[
\mathcal{L}^{-1}\left(\frac{1}{p((p+a)^2+b^2)}\right) = \mathcal{L}^{-1}\left(\frac{1}{a^2+b^2} \left[\frac{1}{p} - \frac{p+a}{(p+a)^2+b^2} - \frac{a}{(p+a)^2+b^2}\right]\right) = \\
= \frac{1}{a^2+b^2} \left[1 - e^{-at} \cos bt - \frac{a}{b} e^{-at} \sin bt\right].
\]
Transfer function

Step response

\[
\mathcal{L}^{-1} \left\{ \frac{1}{a^2 + b^2} \left[ \frac{1}{p} - \frac{p + 2a}{(p + a)^2 + b^2} \right] \right\} = \\
= \frac{1}{a^2 + b^2} \left[ 1 - e^{-at} \cos bt - \frac{a}{b} e^{-at} \sin bt \right]
\]
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3. **Input – output description**

4. **State-space description**
The state-space transfer function is determined again as

\[ H(p) = \frac{Y(p)}{U(p)}, \]

however \( Y(p) \) now depends on state variables.
Let’s have a continuous linear time-invariant system

\[ x'(t) = A x(t) + B u(t), \]
\[ y(t) = C x(t) + D u(t). \]
Transfer function and state-space description

Laplace transform

Using the Laplace transform, we convert the state-space equations to the algebraic form

\begin{align*}
pX(p) - x(0) &= AX(p) + BU(p) \\
Y(p) &= CX(p) + DU(p)
\end{align*}

that we rewrite to

\[(pI - A)X(p) = x(0) + BU(p)\]

and calculate

\[X(p) = (pI - A)^{-1}x(0) + (pI - A)^{-1}BU(p).\]
The transfer function is defined for a zero initial condition $x(0) = 0$. After substituting we get

$$Y(p) = C (pI - A)^{-1} B U(p) + D U(p)$$

$$= \left[ C (pI - A)^{-1} B + D \right] U(p)$$

and transfer function is

$$H(p) = C (pI - A)^{-1} B + D$$
Transfer function and state-space description

\[ H(p) = C(pI - A)^{-1} B + D \]
If it is a pure system that has no direct link from input to output and thus, $D = 0$, then

$$H(p) = C (pI - A)^{-1} B = C \frac{\text{adj}(pI - A)}{\det(pI - A)} B$$
Example (Wheel suspension)

\[ y(t) = x_1(t) \]

\[ x_3(t) \]

\[ u(t) \]

vehicle

damper \( k_d \)
suspension \( k_s \)
tire \( k_t \)
road

The picture shows the vehicle suspension model with coefficients of stiffness \( k_t, k_s \) a \( k_d \).
Transfer function and state-space description – example II

Example (Wheel suspension)

If we apply the equations of motion

\[ Mx'''_3(t) + k_t [x_3(t) - u(t)] - k_s [x_1(t) - x_3(t)] - k_d [x'_1(t) - x'_3(t)] = 0 \]
\[ mx''_1(t) + k_s [x_1(t) - x_3(t)] + k_d [x'_1(t) - x'_3(t)] = 0 \]

Find state-space equations using vector of state variables
\[ \mathbf{x} = [x_1(t), x_2(t), x_3(t), x_4(t)]^T. \]
Find the transfer function \( H(p) = Y(p)/U(p) \), which characterizes the behavior of the vehicle depending on the road surface.
Transfer function and state-space description – example III

Example (Wheel suspension)

We choose

\[ x_1(t) = y(t), \quad x_1'(t) = x_2(t) \]
\[ x_3(t), \quad x_3'(t) = x_4(t) \]

and we get system of equations

\[ x_1'(t) = x_2(t), \]
\[ x_2'(t) = -\frac{k_s}{m} [x_1(t) - x_3(t)] - \frac{k_d}{m} [x_1'(t) - x_3'(t)] \]
\[ x_3'(t) = x_4(t) \]
\[ x_4'(t) = -\frac{k_t}{M} [x_3(t) - u(t)] + \frac{k_s}{M} [x_1(t) - x_3(t)] + \frac{k_d}{M} [x_1'(t) - x_3'(t)] \]
Transfer function and state-space description – example IV

Example (Wheel suspension)

We convert equations to state-space description

\[
\begin{bmatrix}
  x'_1(t) \\
  x'_2(t) \\
  x'_3(t) \\
  x'_4(t)
\end{bmatrix} = \begin{bmatrix}
  0 & 1 & 0 & 0 \\
  -k_s & -k_d & k_s & k_d \\
  -m & -m & m & m \\
  0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
  x_1(t) \\
  x_2(t) \\
  x_3(t) \\
  x_4(t)
\end{bmatrix} + \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  k_t
\end{bmatrix} u(t)
\]

\[ y(t) = \begin{bmatrix}
  1 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
  x_1(t) \\
  x_2(t) \\
  x_3(t) \\
  x_4(t)
\end{bmatrix} .\]
Example (Wheel suspension)

We have state-space matrices in the form

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
\frac{k_s}{m} & \frac{k_d}{m} & \frac{k_s}{m} & \frac{k_d}{m} \\
0 & 0 & \frac{k_s + k_t}{M} & \frac{k_d}{M} \\
\frac{k_s}{M} & \frac{k_d}{M} & \frac{k_s}{M} & \frac{k_d}{M}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 \\
0 \\
0 \\
\frac{k_t}{M}
\end{bmatrix}
\]

and

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0
\end{bmatrix}
\]
Transfer function and state-space description – example VI

Example (Wheel suspension)

and because of $H(p) = C (pI - A)^{-1}B$, i.e.

$$H(p) = \frac{1}{\Delta} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * & * & * & -\Delta_{41} \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_t}{M} \end{bmatrix} =$$

$$= \frac{(k_d p + k_s) k_t}{(M p^2 + k_t)(m p^2 + k_d p + k_s) + m (k_d p + k_s) p^2},$$

where * denotes elements of an inverse matrix that we do not need to calculate for the transfer function $\Delta = \det (pI - A)$. 
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4 State-space description
BIBO stability – bounded input bounded output

The response to a limited input signal must always be limited - the system is BIBO stable.

The system response is a combination

- input signal response
- responses to initial conditions
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   - Marginally stable LTI system

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In the case of the transfer function given by the equation

\[ \frac{d^2}{dt^2} y(t) + 2a \frac{d}{dt} y(t) + (a^2 + b^2) y(t) = u(t) \]

the transfer function poles are complex numbers

\[ p_1 = -a + bi, \]
\[ p_2 = -a - bi. \]

For system stability are critical values \( \Re(p_1) \) a \( \Re(p_2) \). These are complex conjugate poles (others cannot be in LTI systems), and therefore \( \Re(p_1) = \Re(p_2) = -a \), and only value of the parameter \( a \) is important for system stability.
Theorem (Stable continuous-time LTI system)

The real part of all the poles of the transfer function $H(p)$ of the stable system lies in the left part of $p$-plane.

Example (Transfer of 2nd order LTI system – simple poles)

The transfer function of 2nd order LTI system is

$$H(p) = \frac{1}{p^2 + 4p + 3} = \frac{1}{(p + 1)(p + 3)}.$$ 

Transfer function poles $p_1 = -1$ a $p_2 = -3$ lie on the left side of $p$-plane, and therefore system is stable. Note that

$$h(t) = \mathcal{L}^{-1}\{H(p)\} = \mathcal{L}^{-1}\left\{ \frac{k_1}{p + 1} + \frac{k_2}{p + 3} \right\} = \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}.$$
Theorem (Stable continuous-time LTI system)

The real part of all the poles of the transfer function $H(p)$ of the stable system lies in the left part of $p$-plane.
Theorem (Stable continuous-time LTI system)

The real part of all the poles of the transfer function $H(p)$ of the stable system lies in the left part of $p$-plane.

Example (Transfer of 2nd order LTI system – complex conjugate poles)

The transfer function of 2nd order LTI system is

$$H(p) = \frac{1}{p^2 + 4p + 20}.$$ 

Two complex conjugate poles are on the left side of $p$-plane. It is therefore a stable system. Note that

$$h(t) = \mathcal{L}^{-1} \left\{ \frac{4}{4 (p + 2)^2 + 16} \right\} = \frac{1}{4} e^{-2t} \sin(4t).$$
Theorem (Stable continuous-time LTI system)

*The real part of all the poles of the transfer function $H(p)$ of the stable system lies in the left part of $p$-plane.*

\[
p_1 = -2 - i \\
p_2 = -2 + i
\]
Unstable continuous-time LTI system

Theorem (Unstable continuous-time LTI system)

The real part of at least one pole of the transfer function $H(p)$ of the unstable continuous LTI system lies (i) in the right part of $p$-plane, or (ii) has such a system of multiple poles of $H(p)$ on the imaginary axis.

Example (Unstable continuous-time LTI system – real poles)

The transfer function of 2nd order LTI system is

$$H(p) = \frac{1}{p^2 + p - 2} = \frac{1}{(p + 2)(p - 1)}.$$

One of the poles of the transfer function, $p_2 = 1$, lies in the right side of $p$-plane and therefore the system is unstable. Note that

$$h(t) = \mathcal{L}^{-1}\{H(p)\} = \mathcal{L}^{-1}\left\{ \frac{k_1}{p + 2} + \frac{k_2}{p - 1} \right\} = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^t.$$
Theorem (Unstable continuous-time LTI system)

The real part of at least one pole of the transfer function $H(p)$ of the unstable continuous LTI system lies (i) in the right part of $p$-plane, or (ii) has such a system of multiple poles of $H(p)$ on the imaginary axis.

![Diagram showing poles and curve](image.png)
Theorem (Unstable continuous-time LTI system)

The real part of at least one pole of the transfer function $H(p)$ of the unstable continuous LTI system lies (i) in the right part of $p$-plane, or (ii) has such a system of multiple poles of $H(p)$ on the imaginary axis.

Example (Unstable continuous-time LTI system – multiple pole)

The transfer function of 2nd order LTI system is

$$H(p) = \frac{1}{(p^2 + 1)^2}.$$ 

Multiple pole of transfer function $p_\infty = \pm i$ lie on imaginary axis of $p$-plane and therefore system is unstable. Note that

$$h(t) = \mathcal{L}^{-1} \{ H(p) \} = \mathcal{L}^{-1} \left\{ \frac{1}{(p^2 + 1)^2} \right\} = t \sin(t).$$
Unstable continuous-time LTI system

Theorem (Unstable continuous-time LTI system)

The real part of at least one pole of the transfer function $H(p)$ of the unstable continuous LTI system lies (i) in the right part of $p$-plane, or (ii) has such a system of multiple poles of $H(p)$ on the imaginary axis.

$\mathbb{R}(p)$

$\mathbb{I}(p)$

$p_1, p_2 = i$

$p_2, p_3 = -i$

$h(t)$

$t$
Marginally stable continuous-time LTI system

Theorem (Marinally stable continuous-time LTI system)

The real part of the poles of the transfer function $H(p)$ of the marginally stable continuous-time LTI system is equal to zero and the poles are not multiple.

Example (Marginally stable continuous-time LTI system of 1st order – single pole)

The transfer function of 1st order LTI system is

$$H(p) = \frac{1}{p}.$$
Marginally stable continuous-time LTI system

Theorem (Marginally stable continuous-time LTI system)

The real part of the poles of the transfer function $H(p)$ of the marginally stable continuous-time LTI system is equal to zero and the poles are not multiple.
Marginally stable continuous-time LTI system

Theorem (Marginally stable continuous-time LTI system)

The real part of the poles of the transfer function $H(p)$ of the marginally stable continuous-time LTI system is equal to zero and the poles are not multiple.

Example (Marginally stable continuous-time LTI system of 2nd order – complex conjugate pole)

The transfer function of 2nd order LTI system is

$$H(p) = \frac{1}{p^2 + 4}.$$

Note that

$$h(t) = \mathcal{L}^{-1} \{H(p)\} = \mathcal{L}^{-1} \left\{ \frac{1}{2} \frac{2}{p^2 + 2^2} \right\} = \frac{1}{2} \sin(2t).$$
Theorem (Marginally stable continuous-time LTI system)

The real part of the poles of the transfer function $H(p)$ of the marginally stable continuous-time LTI system is equal to zero and the poles are *not multiple*. 

\begin{align*}
p_1 &= i \\
p_2 &= -i
\end{align*}

\[ \Re(p) \quad \Im(p) \]

\[ h(t) \]

\[ t \]
**Stability criterion**

**Summary**

**Stable system**

- \( \lim_{t \to \infty} h(t) = 0 \)
- All poles of the transfer function \( H(p) \) lie in the left half-plane of the complex plane, \( \Re(p_\infty) < 0 \).

**Unstable system**

- \( \lim_{t \to \infty} h(t) = \infty \)
- At least one pole of the transfer function \( H(p) \) lies in the right half-plane of the complex plane, \( \Re(p_\infty) > 0 \) or at least one multiple pole lie on imaginary axis.

**Marginally stable**

- \( \lim_{t \to \infty} h(t) = c \neq 0 \) or does not exist
- At least one **simple** pole lies on imaginary axis and no pole is in the right half-plane of the complex plane. Possible multiple poles lie in the left half-plane.
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Stability of the state-space system

Again we determine the stability on the basis of positions of the poles of the transfer function $H(p)$ in $p$-plane. In the case of state-space description, the transfer function is defined as

$$H(p) = C (p I - A)^{-1} B + D.$$ 

Theorem (State space system stability)

*The matrix $A$ is crucial for system stability, respectively $\det(p I - A)$. Corresponding to the denominator of the transfer function.*

The process is then identical to determining the stability of the system with input–output description.
Example (Transfer function of the state-space LTI system)

The input-output description of a continuous-time LTI system corresponds to the differential equation of the 2nd order

\[
\frac{d^2}{dt^2}y(t) + 4 \frac{d}{dt}y(t) + 20y(t) = u(t).
\]

The transfer function is

\[
H(p) = \frac{1}{p^2 + 4p + 20}.
\]
Example (Transfer function of the state-space LTI system)

When converting to state-space description, we set states as

\[ x_1(t) = y(t), \]
\[ x_2(t) = \frac{d}{dt} y(t), \]

the system matrix \( A \), expression \((pI - A)\) and its determinant equal

\[
A = \begin{bmatrix}
0 & 1 \\
-20 & -4
\end{bmatrix},
\]
\[
pI - A = \begin{bmatrix}
p & -1 \\
20 & p + 4
\end{bmatrix},
\]
\[
det(pI - A) = p^2 + 4p + 20.
\]
Example (Transfer function of the state-space LTI system)

Polynomial $p^2 + 4p + 20$ is a system characteristic polynomial and its roots are the poles of the transfer function.

Further decision on the stability of the state-space system is consistent with the input-output description stability criteria.